# Harmonic Linear Circuits approach using Hilbert transform

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***Abstract*** **– This paper presents an approach to linear circuits using Hilbert transform. The phase shift between current and voltage is to be determined using the proposed method. An important aspect is that the circuit configuration is not known. This is possible by using the Hilbert transform. Also, one can determine the active power, reactive power and apparent power of the circuit. Applications refer strictly to sampled signals, discrete signals.**

***Keywords*** **– phase shift, linear circuits, Hilbert transform, power measurement.**

## 1. Introduction

An important category of electrical circuits is represented by the linear electric circuits operating in constant current mode. Circuits considered consist of RLC components. They are called linear because the current and voltage dependence of these components are linear. Linear electric circuits know particular importance due to their technical applications [5],[6].

This paper presents a method for determining the characteristic impedance of a linear circuit when the input signal and the output signal are known but the circuit structure isn't.

In contrast to the classical method of circuit solving, the proposed method comes with an advantage given by the Hilbert transform. In addition, Hilbert transform provides an equation by which the active and passive components of the circuit are to be determined[2].

Having the components mentioned above one can calculate the phase shift between voltage and current. Also, one can calculate the active, reactive and apparent power in circuit.

The proposed method becomes important because the difficulty of circuit solving remains unchanged no matter how complicated is its structure. An immediate advantage is represented by computing speed and rapidity of the method[3].

For example, two cases are taken into consideration: in the first case the input signal is represented by the sinusoidal constant current and in the second case the input signal is represented by the voltage of the circuit[7].

For both methods is important to make the transition from the continuous time to discrete time.

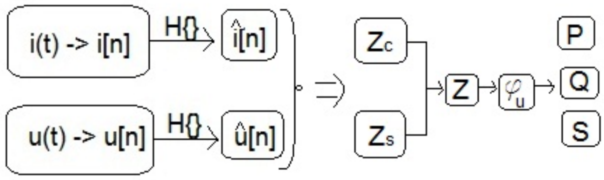



*Figure 1.1.The method steps*

## 2. Theoretical Considerations

Let there be a linear circuit whose configuration is not known.

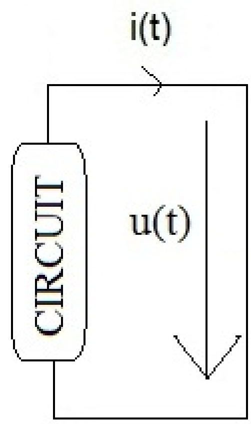



*Figure 2.1.Linear circuit*

### *2.1. Circuit powered by a constant current generator*

For the considered circuit the following is known: the current $i(t)$ and the voltage $u(t)$ [8], [10]:

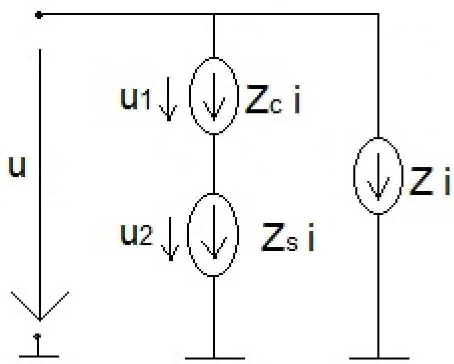



*Figure 2.2.The equivalent circuit*

In continuous time:

The input signal is provided by a constant current generator and is given below:

$$i(t) = I\cos(\omega_0 t + \varphi_i) \qquad (2.1)$$

The initial phase of the input signal was chosen:

$$\varphi_i = 0 \qquad (2.2)$$

Hence, the current through the circuit becomes:

$$i(t) = I\cos(\omega_0 t) \qquad (2.3)$$

Hilbert transform of the input signal becomes [1]:

$$TH\{i(t)\} = \hat{i}(t) = I\sin(\omega_0 t) \qquad (2.4)$$

The output signal is given below:

$$u(t) = U\cos(\omega_0 t + \varphi_u) \qquad (2.5)$$

It can be written like this:

$$\begin{aligned} u(t) &= U\cos(\omega_0 t)\cos(\varphi_u) - U\sin(\omega_0 t)\sin(\varphi_u) = \\ &= Z\cos(\varphi_u) I\cos(\omega_0 t) - Z\sin(\varphi_u) I\sin(\omega_0 t) = \\ &= Z\cos(\varphi_u) i(t) - Z\sin(\varphi_u)\hat{i}(t) \end{aligned} \qquad (2.6)$$

where $Z$ is the characteristic impedance of the circuit, as shown in Figure (2.1), and $\varphi_u$ represents the initial voltage phase.

The following notations are used:

$$Z_c = Z\cos(\varphi_u) \qquad (2.7)$$

$$Z_s = -Z\sin(\varphi_u) \qquad (2.8)$$

By replacing (2.7) and (2.8) in the last equation of (2.6), the first equation of the system is revealed (2.9). The unknown data of the system are $Z_c$ and $Z_s$.

$$\begin{aligned} u(t) &= Zi(t) = \\ &= Z_c i(t) + Z_s \hat{i}(t) \end{aligned} \qquad (2.9)$$

$$u(t) = u_1(t) + u_2(t) \qquad (2.10)$$

By applying the Hilbert transform to the first equation of the system, the second equation of the system is obtained:

$$TH\{u(t)\} = \hat{u}(t) = Z_c \hat{i}(t) - Z_s i(t) \qquad (2.11)$$

In (2.11) Hilbert transform property was used stating that: if Hilbert transform is applied twice consecutively to a signal, the original signal denied is obtained [6-7]:

$$\widehat{[\hat{i}(t)]} = -i(t) \qquad (2.12)$$

After calculations, using (2.7) and (2.8),is obtained:

$$Z^2 = Z_c^2 + Z_s^2 \qquad (2.13)$$

$Z$ can be also calculated from (2.9):

$$Z = Z_c + Z_s \frac{\hat{i}(t)}{i(t)} \qquad (2.14)$$

Knowing that:

$$\begin{aligned} \underline{I} &\leftrightarrow I\cos(\omega_0 t) = i(t) \\ \underline{U} &\leftrightarrow U\cos(\omega_0 t) = u(t) \end{aligned} \qquad (2.15)$$

$$j\underline{I} \leftrightarrow I\cos(\omega_0 t + \frac{\pi}{2}) = -H\{i(t)\} \qquad (2.16)$$

$$-j\underline{I} \leftrightarrow I\cos(\omega_0 t - \frac{\pi}{2}) = H\{i(t)\} \qquad (2.17)$$

Equation (2.9) can be written like this:

$$\underline{U} = Z_c \underline{I} + Z_s(-j\underline{I}) \qquad (2.18)$$

$$\underline{U} = Z_c \underline{I} - jZ_s \underline{I} \qquad (2.19)$$

For a better view, the image below shows the phase diagram:

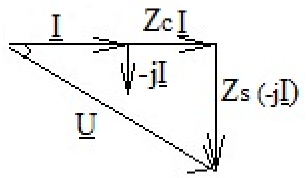



*Figure 2.3.The phase diagram*

Now, one can determine the active component of the circuit $Z_c$ and the reactive component of the circuit $Z_s$. This is possible with the help of computational short-cuts starting with (2.9) and (2.11).

For the beginning $Z_s$ is obtained:

$$u(t)\hat{i}(t) = Z_c i(t)\hat{i}(t) + Z_s \hat{i}^2(t) \qquad (2.20)$$

$$\hat{u}(t) i(t) = Z_c \hat{i}(t) i(t) - Z_s i^2(t) \qquad (2.21)$$

$$u(t)\hat{i}(t) - \hat{u}(t) i(t) = Z_s[\hat{i}^2(t) + i^2(t)] \qquad (2.22)$$

$$Z_s = \frac{u(t)\hat{i}(t) - \hat{u}(t) i(t)}{\hat{i}^2(t) + i^2(t)} \qquad (2.23)$$

Similarly $Z_c$ is to be determined:

$$u(t) i(t) = Z_c i^2(t) + Z_s \hat{i}(t) i(t) \qquad (2.24)$$

$$\hat{u}(t)\hat{i}(t) = Z_c \hat{i}^2(t) - Z_s i(t)\hat{i}(t) \qquad (2.25)$$

$$u(t) i(t) + \hat{u}(t)\hat{i}(t) = Z_c[i^2(t) + \hat{i}^2(t)] \qquad (2.26)$$

$$Z_c = \frac{u(t) i(t) + \hat{u}(t)\hat{i}(t)}{i^2(t) + \hat{i}^2(t)} \qquad (2.27)$$

Next, the impedance of the circuit can be calculated with (2.13):

$$Z = \sqrt{Z_c^2 + Z_s^2} \quad (2.28)$$

The angle $\varphi_u$ can be determined of (2.7) or (2.8):

$$\varphi_u = \arccos \frac{Z_c}{Z} \quad (2.29)$$

$$\varphi_u = -\arcsin \frac{Z_s}{Z} \quad (2.30)$$

Considering that the input signal initial phase is zero, $\varphi_u$ also represents the phase shift between voltage and current. [A1]

$$\varphi = \varphi_u - \varphi_i \quad (2.31)$$

$$\varphi = \varphi_u \quad (2.32)$$

*2.2. Circuit powered by a constant voltage source*

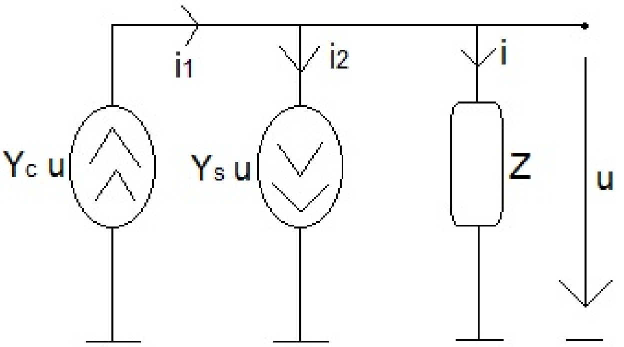



*Figure 2.4.The equivalent circuit*

In continuous time [8], [10]:

The input signal is given below:

$$u(t) = U \cos(\omega_0 t + \varphi_u) \quad (2.33)$$

The initial phase of the input signal was chosen zero.

$$\varphi_u = 0 \quad (2.34)$$

The voltage of the circuit becomes:

$$u(t) = U \cos(\omega_0 t) \quad (2.35)$$

Hilbert transform of the input signal becomes:

$$TH\{u(t)\} = \hat{u}(t) = U \sin(\omega_0 t) \quad (2.36)$$

The current through the circuit is given below:

$$i(t) = I \cos(\omega_0 t + \varphi_i) \quad (2.37)$$

It can be written like this:

$$\begin{aligned} i(t) &= I \cos(\omega_0 t) \cos(\varphi_i) - I \sin(\omega_0 t) \sin(\varphi_i) = \\ &= Y \cos(\varphi_i) U \cos(\omega_0 t) - Y \sin(\varphi_i) U \sin(\omega_0 t) = \\ &= Y \cos(\varphi_i) u(t) - Y \sin(\varphi_i) \hat{u}(t) \end{aligned} \quad (2.38)$$

where $Y$ represents the characteristic admittance of the circuit, as shown in Figure (3.4), and $\varphi_i$ represents the current initial phase.

The following notations are used:

$$Y_c = Y \cos(\varphi_i) \quad (2.39)$$

$$Y_s = -Y \sin(\varphi_i) \quad (2.40)$$

By replacing (2.32) and (2.33) in the last equation of (2.31), the first equation of the system is obtained (2.35). The unknown data of the system are $Y_c$ and $Y_s$ .

$$i(t) = Yu(t) = Y_c u(t) + Y_s \hat{u}(t) \quad (2.41)$$

$$i(t) = i_1(t) + i_2(t) \quad (2.42)$$

After calculations, using (2.32) and (2.33), $Y$ is obtained:

$$Y^2 = Y_c^2 + Y_s^2 \quad (2.43)$$

Also, $Y$ can be obtained from (2.35'):

$$Y = Y_c + Y_s \frac{\hat{u}(t)}{u(t)} \quad (2.44)$$

By applying the Hilbert transform to the first equation of the system, the second equation of the system is obtained:

$$TH\{i(t)\} = \hat{i}(t) = Y_c \hat{u}(t) - Y_s u(t) \quad (2.45)$$

Knowing that:

$$\underline{U} \leftrightarrow U \cos(\omega_0 t) = u(t) \quad (2.46)$$

$$\underline{jU} \leftrightarrow U \cos(\omega_0 t + \frac{\pi}{2}) = -H\{u(t)\} \quad (2.47)$$

$$-\underline{jU} \leftrightarrow U \cos(\omega_0 t - \frac{\pi}{2}) = H\{u(t)\} \quad (2.48)$$

Equation (2.41) can be written like this:

$$\underline{I} = Y_c \underline{U} + Y_s (-j\underline{U}) \quad (2.49)$$

$$\underline{I} = Y_c \underline{U} - jY_s \underline{U} \quad (2.50)$$

For a better view, the image below shows the phase diagram:

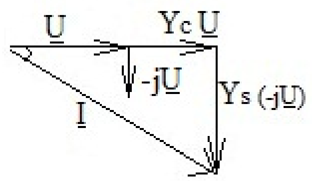



*Figure 2.5.The phase diagram*

Is intended to determine the active component of the circuit $Y_c$ and the reactive component of the circuit $Y_s$. This is possible with the help of computational short-cuts starting with (2.41) and (2.45).

For the beginning $Y_s$ is obtained:

$$i(t)\hat{u}(t) = Y_c u(t)\hat{u}(t) + Y_s \hat{u}^2(t) \quad (2.37)$$

$$\hat{i}(t)u(t) = Y_c \hat{u}(t)u(t) - Y_s u^2(t) \quad (2.38)$$

$$i(t)\hat{u}(t) - \hat{i}(t)u(t) = Y_s[\hat{u}^2(t) + u^2(t)] \quad (2.39)$$

$$Y_s = \frac{i(t)\hat{u}(t) - \hat{i}(t)u(t)}{\hat{u}^2(t) + u^2(t)} \quad (2.40)$$

Similarly, $Y_c$ is to be determined:

$$u(t)i(t) = Y_c u^2(t) + Y_s \hat{u}(t)u(t) \quad (2.41)$$

$$\hat{u}(t)\hat{i}(t) = Y_c \hat{u}^2(t) - Y_s u(t)\hat{u}(t) \quad (2.42)$$

$$u(t)i(t) + \hat{u}(t)\hat{i}(t) = Y_c[u^2(t) + \hat{u}^2(t)] \quad (2.43)$$

$$Y_c = \frac{u(t)i(t) + \hat{u}(t)\hat{i}(t)}{u^2(t) + \hat{u}^2(t)} \quad (2.44)$$

## 3. Applications

### *3.1. Circuit powered by a constant current generator*

Let there be a linear circuit with an unknown structure powered by a constant current source as can be seen in Figure 3.1:

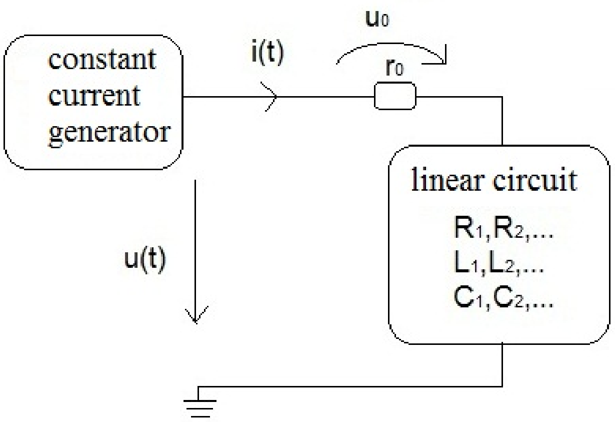



*Figure 3.1.circuit application 1*

Knowing the resistance $r_0$, the current *i(t)* thru the circuit (Figure 3.1.) is calculated.The voltage across the resistor is measured and the current in the circuit is calculated [4]:

$$i(t) = \frac{u_0(t)}{r_0} \quad (3.1)$$

A voltmeter was used to measure the voltage of the circuit.

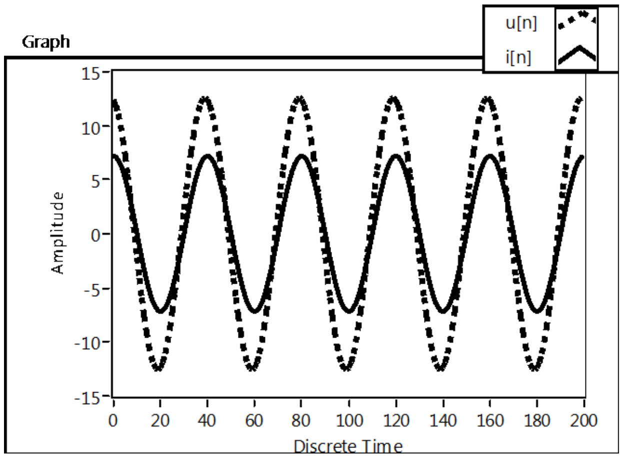



*Figure 3.2.The input and the output signal*

In discrete time:

By discretization of the input signal the discrete signal $i[n]$ is obtained:

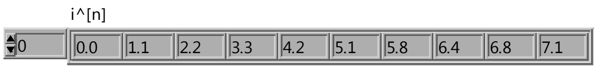
$$i[n] = \{i[0], i[1], ..., i[n-1]\} \quad (3.2)$$


Then, the output signal is measured using a digital voltmeter and $u[n]$ is obtained:

$$u[n] = \{u[0], u[1], ..., u[n-1]\} \quad (3.3)$$


By applying the Hilbert transform the Hilbert conjugates are obtained:

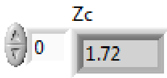
$$THD\{i[n]\} = \hat{i}[n] \quad (3.4)$$



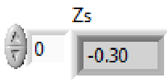

$$THD\{u[n]\} = \hat{u}[n] \quad (3.5)$$


The active and the reactive components of the circuit are to be determined:

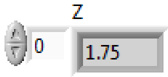
$$Z_c = \frac{u[n]i[n] + \hat{u}[n]\hat{i}[n]}{i^2[n] + \hat{i}^2[n]} \quad (3.6)$$

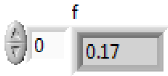



$$Z_s = \frac{u[n]\hat{i}[n] - \hat{u}[n]i[n]}{\hat{i}^2[n] + i^2[n]} \quad (3.7)$$



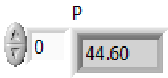
The impedance of the circuit, $Z$, is calculated with (2.28) and the voltage phase is determined by (2.29) or (2.30).



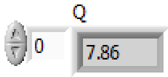




Also, the active power $P$, the reactive power $Q$ and the apparent power $S$ can be revealed [9]:

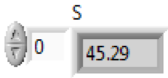
$$P = U_{ef} I_{ef} \cos\varphi = I^2{}_{ef} Z_c \quad [W] \qquad (3.8)$$




$$Q = U_{ef} I_{ef} \sin\varphi = -I^2{}_{ef} Z_s \quad [VAR] \qquad (3.9)$$




$$S = U_{ef} I_{ef} = \sqrt{P^2 + Q^2} \quad [VA] \qquad (3.10)$$




In continuous time:

Taking into consideration that the power can be expressed as a mean value $(m.v.)$, it can be calculated by the following method [2-3]:

$$P = m.v.\{i(t)u(t)\} \qquad (3.11)$$

but $u(t) = Z_c i(t) + Z_s \hat{i}(t)$

$$P = m.v\{i(t)[Z_c i(t) + Z_s \hat{i}(t)]\} =$$

$$= m.v.\{Z_c i^2(t) + Z_s i(t)\hat{i}(t)\} = \qquad (3.12)$$

$$= m.v.\{Z_c i^2(t)\} + m.v.\{Z_s i(t)\hat{i}(t)\}$$

It is well known that $m.v.\{Z_s i(t)\hat{i}(t)\} = 0$.

$$P = m.v.\{Z_c i^2(t)\} \qquad (3.13)$$

$$Q = m.v.\{i(t)\hat{u}(t)\} \qquad (3.14)$$

but $\hat{u}(t) = Z_c \hat{i}(t) - Z_s i(t)$:

$$Q = m.v.\{i(t)[Z_c \hat{i}(t) - Z_s i(t)]\} =$$

$$= m.v.\{Z_c i(t)\hat{i}(t)\} - m.v.\{Z_s i^2(t)]\} \qquad (3.15)$$

It's well known that $m.v.\{Z_c i(t)\hat{i}(t)\} = 0$.

$$Q = m.v.\{Z_s i^2(t)]\} \qquad (3.16)$$

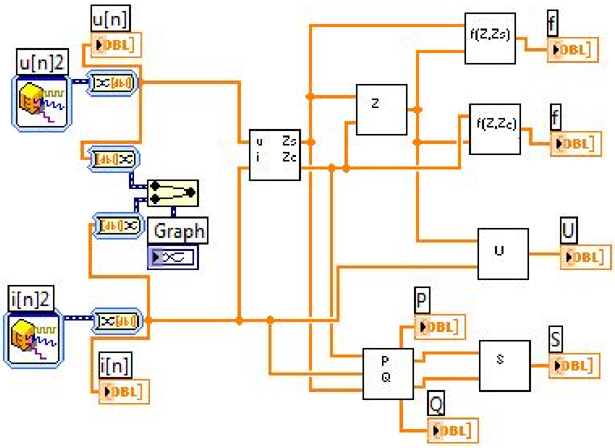



*Figure 3.3.LabView application 1*

*3.2. Circuit powered by a constant voltage source*

Let there be a linear circuit with an unknown structure powered by a constant voltage source as shown in Figure 3.4:

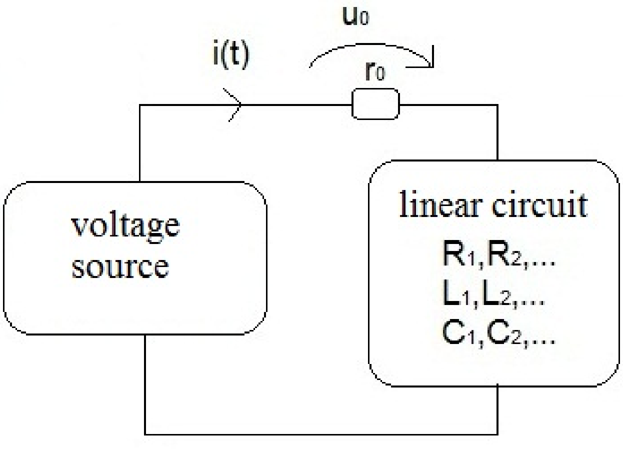



*Figure 3.4.Application 2 circuit*

For considered circuit, the voltage *u(t)* is measured by the parallel connection of a digital voltmeter. The signal *i(t)* is measured with the resistance $r_0$.

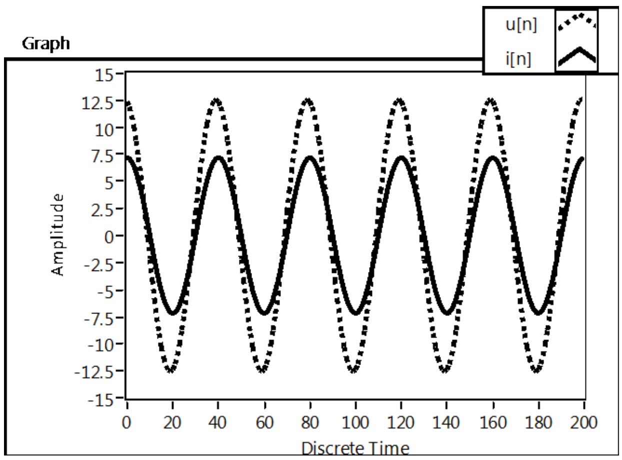



*Figure 3.5.The input and the output signals*

In discrete time:

Discrete signal $u[n]$ represents the circuit input:

| u[n] | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 12.4 | 11.9 | 11.1 | 10 | 8.7 | 7.2 | 5.5 | 3.7 | 1.8 | -0.2 |

The circuit current is represented by $i[n]$:

| i[n] | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 7.2 | 7.1 | 6.8 | 6.4 | 5.8 | 5.1 | 4.2 | 3.3 | 2.2 | 1.1 |

By applying the Hilbert transform the Hilbert conjugates are obtained:

$$THD\{u[n]\} = \hat{u}[n] \qquad (3.17)$$

| u^[n] | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 2.2 | 4.1 | 5.9 | 7.6 | 9.0 | 10.3 | 11.3 | 12.0 | 12.5 | 12.6 |

$$THD\{i[n]\} = \hat{i}[n] \qquad (3.18)$$

| i^[n] | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0.0 | 1.1 | 2.2 | 3.3 | 4.2 | 5.1 | 5.8 | 6.4 | 6.8 | 7.1 |

The active and the reactive components of the circuit are to be determined:

$$Y_c = \frac{i[n]u[n] + \hat{u}[n]\hat{i}[n]}{u^2[n] + \hat{u}^2[n]} \qquad (3.19)$$

Yc: 0 | 0.56

$$Y_s = \frac{i[n]\hat{u}[n] - \hat{i}[n]u[n]}{\hat{u}^2[n] + u^2[n]} \qquad (3.20)$$

Ys: 0 | 0.10

The admittance of the circuit, $Y$, is calculated with (2.43) and the voltage phase is determined by (2.39) or (2.40).

Y: 0 | 0.57

f: 0 | 0.17

Also, the active power $P$, the reactive power $Q$ and the apparent power $S$ can be revealed [9]:

$$P = U_{ef} I_{ef} \cos\varphi = U^2{}_{ef} Y_C \quad [W] \qquad (3.21)$$

P: 0 | 44.6

$$Q = U_{ef} I_{ef} \sin\varphi = -U^2{}_{ef} Y_S \quad [VAR] \qquad (3.22)$$

Q: 0 | 7.9

$$S = U_{ef} I_{ef} = \sqrt{P^2 + Q^2} \; [VA] \qquad (3.23)$$

S: 0 | 45.3

*In continuous time:*

Taking into consideration that the power can be expressed as a mean value $(m.v.)$, it can be calculated by the following method:

$$P = val.med.\{i[t]u[t]\} \qquad (3.24)$$

but $i(t) = Yu(t) = Y_c u(t) + Y_s \hat{u}(t)$

$$P = m.v.\{[Y_c u(t) + Y_s \hat{u}(t)]u(t)\} =$$
$$= m.v.\{Y_c u^2(t) + Y_s u(t)\hat{u}(t)\} = \qquad (3.25)$$
$$= m.v.\{Y_c u^2(t)\} + m.v.\{Y_s u(t)\hat{u}(t)\}$$

It is well known that $m.v.\{Y_s u(t)\hat{u}(t)\} = 0$

$$P = m.v.\{Y_c u^2(t)\} \qquad (3.26)$$

$$Q = m.v.\{i(t)\hat{u}(t)\} \qquad (3.27)$$

$$Q = m.v.\{\hat{u}(t)[Y_c u(t) + Y_s \hat{u}(t)\} =$$
$$= m.v.\{Y_c \hat{u}(t)u[(t)\} + m.v.\{Y_s \hat{u}^2(t)\} \qquad (3.28)$$

but $m.v.\{Y_c \hat{u}(t)u(t)\} = 0$

$$Q = m.v.\{Y_s \hat{u}^2(t)\} \qquad (3.29)$$

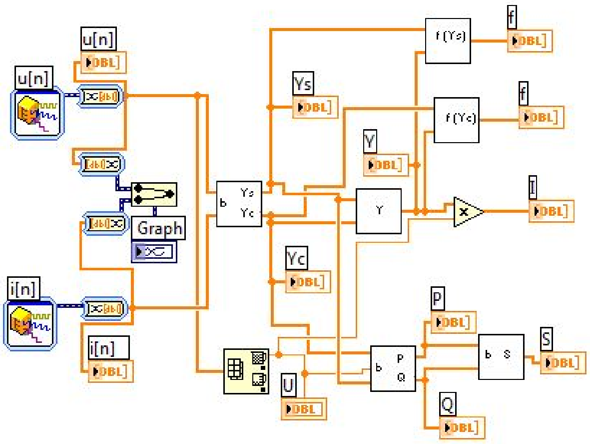



*Figure 3.6.LabView application 2*

## 4. Conclusion

In this paper were presented two ways of determining the phase difference between voltage and current without knowing the circuit configuration. This was possible using the Hilbert transform. By applying the Hilbert transform a new equation was obtained. Having the new formed system it was easy to determine the unknown components: the active component and the passive component of the circuit. In the first presented application, the impedance of the circuit and then the voltage phase was calculated. Because the initial phase of current was chosen zero the initial voltage phase became the phase difference between the two signals – current and voltage. In the second application were calculated equivalent admittance and the current phase through the circuit. This time the initial voltage phase was chosen equal to zero, so the initial current phase became the phase difference between the two signals.

In addition, the proposed method offers the possibility for determining the powers in the circuit: active, reactive and apparent. The powers in the circuit were determined with both method presented. It was possible to make a comparison of the results obtained and a recheck.

The method refers restrictedly to sampled signals, discrete signals and is to be applied to linear circuits. The input signal is a harmonic signal. To illustrate the method, a sinusoidal signal supplying a circuit was chosen whose structure was not known. At the circuit output was obtained a sinusoidal signal delayed from the first signal by angle .

The advantage of the presented method is given by the circuit structure lack of influence on the method. It was observed that a complex or a simple configuration does not influence the proposed method steps. It also does not increase the time resolution depending on the circuit structure. This latter statement makes the method important because it gives calculation speed and rapid solving solution.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