Some Typical Examples of the Application of Didactic Principle of Polyformism in the Teaching of Mathematics

Djoko G. Markovic [1]

[1] *University of Montenegro, Faculty of philosophy in Niksic, Montenegro*

Abstract: **Teaching principles are the subject of research in light of didactic polyform principle and their positive correlative links, which they contribute to breaking of formalism in Mathematics Teaching for realization of the most important teaching goals, i.e. making the process more active and more dynamic and adopting deductive methods as the most significant way of thinking, complete acquisition proficiency and its applying to various mathematical and other situations.**

This work concerned with the Mathematics Teaching and especially the principle of polyform, introduced with geometrical-arithmetical-algebraic interpretation in function of dynamic teaching process with tendency of continuous offering of occasions to pupils for original and creative thinking in addition to development of habits of independent polyform thinking, critical evaluation and reasonable generalization.

Keywords: **Principle of polyformism, geometrical polyformism, combine polyformism, methodological innovations, breaking of formalism.**

1. Didactic principles of evidence and polyformism

The principle of evidence is one of the most commonly mentioned didactic principles in the history of teaching. Its basic demand consists of the fact that teaching is designed in such a way that students make clear notions based on the direct, live perception of the studied phenomena or subjects, that is the appearances of the objective world and their immediate polyform presentation [2].

Ian Comenius`s golden rule of teaching – the principle of evidence was treated as more important than any other foundamental principle of teaching in the early phase of development of didactics.

Modern didactics, unlike those first notions of Comenius, Rousseau, Pestalozzi, Disterweg, Ushinski and others who relied upon the sensualistic gnoseology of Bacon and Locke, bases the principle of evidence on the modern knowledge of contemporary psychology - that perceptions of the outer world are an important source of knowledge and that vivid observation relies upon abstract thinking. It is also based on the fact that practice is the basic criteria of truthfulness of knowledge [4]. What should be particularly emphasized is that the principle of evidence is not its own purpose, that means that the introduction of concrete objects presents only the starting thread in the process of learning - as a starting point on the way of acquiring knowledge. It would be wrong to assume that the evidence is only the way from concrete to abstract thinking, even though it is only an important link in the chain of learning, in which the important generalizations come in the end. This conclusion is derived from Arnchaim`s proofs that abstract opinion is formed even on the level of perception.

In modern teaching of Mathematics the principle of evidence occupies a significant place with its main point of support in the psychological law of acquiring knowledge, which states that there is nothing in the mind that hasn`t previously been in the senses in some way [3] .

Didactic principle of polyformism, according to what I have managed to see in thirty years of teaching mathematics in high schools, cannot be found in the literature of mathematics teaching as a didactic particularity and, even if it has been applied somewhere than it was done intuitively, and occasionally in mathematics teaching in primary and secondary schools [7].

The essence of this significant didactic principle consists in the permanent insisting on the integral insight into diverse approaches of understanding and notions of the taught phenomena. Its exploitation in practice requires from the teacher an excellent knowledge and the skill of application of the most versatile expertise in methodological possibilities and it induces an intensive brain activity of the students expressed through their hard and devoted work and higher motivation.

The efficiency of the principle of polyformism is based on the evident psychological fact that the changes and diversity in teaching make it fresh and interesting, whereas monotony induces weaker interest and thus passivity and boredom as well. This is the reason why the principle of polyformism should play a universal role in teaching of mathematics which could be seen through the enrichment of teaching with different contents, means, procedures and methods. When it comes to the contents then it is referred to the choice of such tasks which enable a larger number of different approaches while solving them and the use of the evident means.

However, the organization of such classes demands the adequate application of polyform methodological forms and methodological particularities of teaching, that is its variations during the same lesson. Methodological forms and particularities planned and applied by the teacher during the process of teaching are based on the pulsing of didactic principles done in the right timing, which can be seen in their simultaneous polyform-cohesive acting, that is integral, dialectic unity.

It is unimaginable to talk about any segment of modern dialectics without leaning about the results and facts achieved by modern psychology and didactics. For this reason it is very worrying why the principle of polyformism does not occupy a dominant position in methodology of teaching mathematics, not disputed by the psychological and didactical theories. I would said that these two sciences, in certain, though implicit, way, leave the possibility for such a multiplex principle, which will, through the presentation of complementary opposed groupings, thoroughly and didactically unite these didactic principles of modern teaching of mathematics. In view of the above particularities, the principle of polyforism represents a universal didactic principle, whose gnoseological basis is identical to the principle of permanence, the law of negation of negation, by which the principle of polyformism acquires the features of a dialectical law. Since the principle of polyformism covers all didactic principles, it raises this principle to the throne of universality.

According to the analyses of the R. Ainhaim`s views about the visual thinking and the studies of ideas of L.Vigotski on the relationship of thinking and speaking observed in the sense of integrality of M. Marjanovic`s synthesis on the three-component notion, along with rich personal experience in direct teaching of mathematics in high schools and modern didactic tendencies, based on the laws of dialectics, I have come to a conclusion that the combination of the verbal - textual and illustrative - demonstrative method provides incredible possibilities for the effective application of the method of polyformism.

This should be particularly emphasized when it comes to its application as content components of teaching, or more precisely to the geometrical polyform of "the Shariginovsky type"[5].

The answer to the question: "Why is it so?" lies in the above stated facts, as well as in the fact that the visual thinking (thinking in pictures) has the quality of general meaning, that is the integrality that causes the known effects of "got it" experience, which are boosted by different demonstrations of the same example with the appropriate methodological particularities of the illustrative-demonstrative method in combination with the particularities of the verbally-textual teaching method. This is the simplest way of presenting the given tasks and problems, for example with pictogram writing, which creates easier formation of Ideogram notions of mental images related to the symbolism and examples, I would say with "got it" perception of the three-component structure of the notion, the types of polyform principles of evidence [6].

Complete methodology of mathematics teaching observed in the light of the principle of polyform is based on the fact which can be considered as a didactic axiom, that picturesque interpretations of the teachers are given as multiple evidence, which offers a new dimension to the teaching of mathematics. In that way, even the didactic method of evidence appears here as a polyform character, which is reflected student`s diversified perception of the same objects, phenomena, or their images, under, the teacher`s systematic guidance in tracking the route of the direct acquiring of mathematical knowledge.

For all of these reasons it is my opinion that the polyform application of the principle of evidence, included in the principle of diversity, presents exactly a didactic-dialectic law, or the principle universalis, for which all the great pedagogues like Comenius, Pestalozzi, Disterweg, Ushinski, Tolstoy and many others searched for.

2. Some rarity examples of the principle of polyformism with the supplement of the innovative, historically-reconstructive elements

Here are two well-known problems which can be solved in different ways, but with the simple use of power they can be solved more elegantly as two unusual polyform reconstructive stories in the development of mathematical ideas .

1. Given two segments a and b, construct the segment $x = \sqrt{a \cdot b}$ [8].

The problem can be solved in several ways:

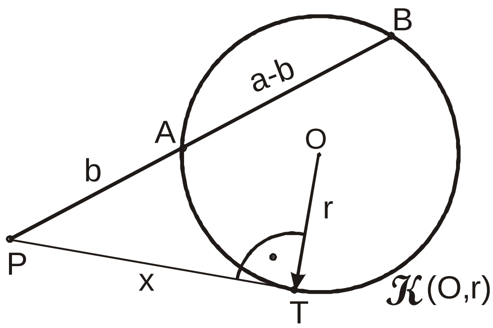
a) If we take that the segment x is the altitude of the rectangular triangle, then *a* and *b* are normal projections of their cathouses on the hypotenuse. Trivial construction, proof and discussion follow from this.

b) We can take that the sought segment x is the cathetus of the rectangular triangle, then bigger than the given segments (it can be segment a) hypotenuse, and the smaller segment b is the normal projection of the sought cathetus x on the hypotenuse. Even in this case construction, proof and the discussion are trivial.

However, the sought geometric mean of the segments a and b can be constructed on the third way by using the power of point in relation to the circuit.

c) In the arbitrary circuit $\mathcal{K}$(O,r) let us draw a cutting line in such a way that the chord on it is AB =a–b and transfer the segment AP = b (picture 1). Tangent segment PT = x is the sought segment. From the property of the point P in relation to the circum ference $\mathcal{K}$(O,r) it follows that $PA \cdot PB = PT^2$, i.e. $b \cdot a = x^2$, that is: $x = \sqrt{a \cdot b}$.

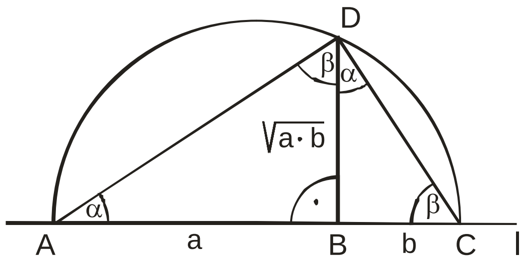
This means that the problem is reduced to the



Picture 1.

construction of the tangent segment PT = x on the circuit $\mathcal{K}$(O,r), which is trivial. The problem is possible if $r \geq \frac{a-b}{2}$.

d) In case a) the proof is usually obtained on the basis of similarities of the rectangular triangles



Picture 2.

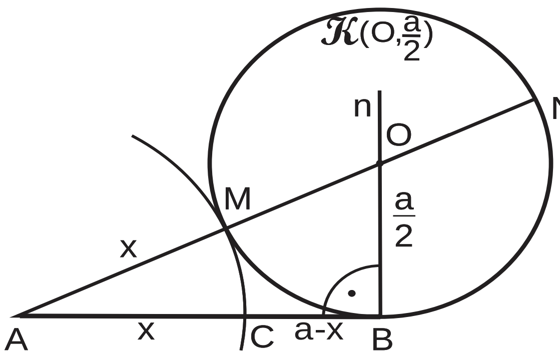
ΔABD and ΔBDC (picture2). However, the proof can be performed by using trigonometry. As $(tg\alpha = \frac{DB}{a}$ and $(tg\alpha = \frac{b}{DB})$ $\Rightarrow$ $DB^2 = a \cdot b \Rightarrow DB = \sqrt{a \cdot b}$.

2) In the given segment AB = a perform the golden section (sectio aurea) [8].

Analyses: To divide the segment on the golden section, means to divide it on two parts, in such a way that its bigger part stands in relation to its smaller part in the same ratio as the whole segment to its bigger part x. If the bigger part is x, then the smaller part is a–x and we have the proportion x**:**(a–x)=a**:**x, or $x^2 = a\cdot(a-x)$, which can be written as $x\cdot(x+a) = a^2$. So the construction can be done with the help of similarity, that is the power point in relation to the circumference.

Construction: From any terminal point of the segment AB, for example point B raise the normal n (picture 3). Transfer to it the segment $BO = \frac{a}{2}$ and draw the circumference $\mathcal{K}\left(O, \frac{a}{2}\right)$, then draw the straight line (AO), and we will get the segment AM =x, because from the power point A in relation to that constructed circle $x\cdot(x+a)= a^2$, that is the imposed condition is fulfilled. Proof follows from the construction, that is $AM\cdot AN=AB^2 \Rightarrow$ $\Rightarrow x\cdot(x+a)=a^2$.

Discussion: The problem has one solution. Here is the construction of the golden section of the



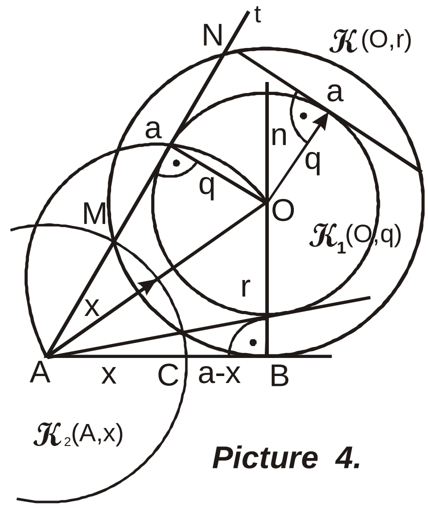
segment AB which can be done in a more general way, which I haven`t seen in the methodological literature and therefore can consider it as new.

Analyses is completely identical to the previous one.

Construction:

From any the two terminal points of the segment AB, for example point B we raise the normal n (picture 4), choose a point on it O and draw the circumference $\mathcal{K}(O,r)$, where $r > \frac{a}{2}$. We construct the circumrerence $\mathcal{K}_1(O,q)$ which is the geometrical place of the chord middle with the length a of the circumference $\mathcal{K}(O,r)$. From point A construct the chord t on the circumference $\mathcal{K}_1(O,q)$. It cuts the circumference $\mathcal{K}(O,r)$ at points M and N, whereby MN = a. The sought point C is obtained at the intersection of the circumference $\mathcal{K}_2(A,AM)$ and segment AB. Proof follows from the construction, that is: $AM\cdot AN = AB^2 \Rightarrow x\cdot(x+a) = a^2$.

When dealing with the problem in which is required to perform the golden section, then it is a right opportunity to tell the story from the history of mathematics teaching, about Pithagoreans and their teaching. The trade mark of Pithagoreans was a regular five-arms star, which is generated with diagonals of the regular pentagon. Here the students sould be reminded of the properties of the diagonals of the regular pentagon, i.e. that they are cut by the golden section. The students, especially the high-school ones, can worke out the task of proving that a side of the regular decagon and radius of the circumference drawn around it stand in the ratio r**:**a = a**:**(r–a) of the golden section. If one bears this in mind then it is quite clear that the previous problem can be solved in different ways, but the one suggested in a) would be the simplest one.



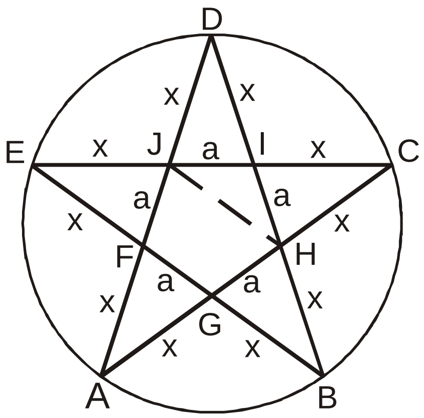
Picture 4.

In modern geometry textbooks, that is mathematics for primary and secondary schools, only one way of construction of the regular pentagon is presented, with the given radius of the circumscribed, circumference, and usually without proof.

Pithagoreans did not accidentally take regular star with five arms, generated by diagonals of the regular pentagon as a sign of their own recognition. It is evident that they did so because a person needed to have solid mathematical knowledge in order to be able to construct it. Before doing the construction of the regular pentagon with a given side a here`s one elementary problem of proving.

Prove that each of the five diagonals of the regular pentagon which form Pithagorean star five-arm star (picture5) divides the other diagonals by cutting them according to the golden section.

Proof: From the similarity



Picture 5.

$\Delta CEG \sim \Delta CJH$ follows CE:CJ=CG:CH.

As CH=JE=CI=x, a CG=CJ=IE=a+x, then CE:CJ=CJ:JE that is (JE+CJ):CJ=CJ:JE. So, the segment AD divides the segment CE by the golden section. In an analogous way proof is performed for the other diagonals of the pentagon. From the previous proofts it is easy to notice that:

(JE+CJ):CJ=CJ:JE $\Rightarrow \dfrac{JE+CJ}{CJ}=\dfrac{EI}{JE} \Rightarrow \quad \Rightarrow$

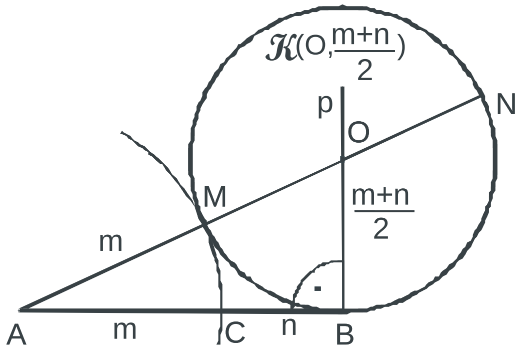
$\dfrac{JE+CJ}{CJ}-\dfrac{CJ}{CJ}=\dfrac{EI}{JE}-\dfrac{JE}{JE} \Rightarrow$

$\Rightarrow \quad \dfrac{JE}{CJ}=\dfrac{EI-JE}{JE} \quad \Rightarrow \quad \dfrac{JE}{CJ}=\dfrac{JI}{JE}$, i.e.

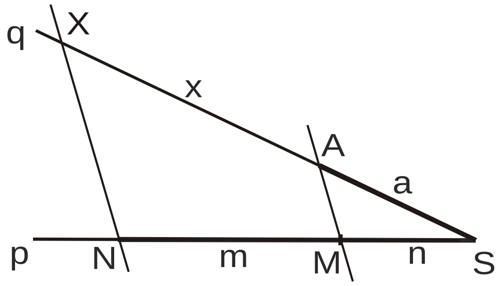
$\dfrac{EJ}{EI}=\dfrac{JI}{EJ}$, so x:(a+x)=a:x,

rpresents the analyses of the problem which demands the construction of a regular pentagon if one of its sides is given.

Construction: If a side a of the regular pentagon is given it is easy to construct, by using the previous analyses and picture 5, a diagonal of the regular pentagon ABCDE with the length $a+2\cdot x$, that is the segment whose measurable number has the length, and in the same way the wanted pentagon too. First on the arbitrary segment AB we will construct point C which divides it on the golden section (picture 6). It is easy to see that $a<x$ and n<m and see that the relation is valid



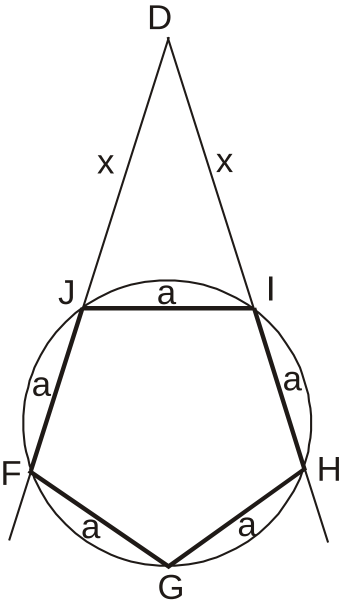
Picture 6.



Picture 7.

$\dfrac{a}{x}=\dfrac{n}{m}$. As a, m i n are three measurable numbers with the lengths of the given segments, the fourth segment, whose measurable number has the length x, can be easily constructed with the use of the Talos`s theorem. On the arbitrary straight lines p i q which intersect in the point S we apply the lengths of the segments n, m and a, as in the picture 7. The segment AX is obtained when through point N a straight line is constructed which is parallel to the straight line MA and which intersects the straight line q in the point X. Now it`s easy to construct a wanted pentagon IHGFJ.

Since the segments with the length a, are given x ΔDJI along with the apexes F, H and G (see picture 8). Since we construct ΔDJI



Picture 8.

on the straight lines DJ and DI from those sides where there isn`t point D points F and H are determined so that JF=IH=a. The apex G is obtained in the intersection of the circuits $\mathcal{K}$(F,a) and $\mathcal{K}$(H,a). Pithagoreans were familiar with the construction of the regular pentagon when a side is given. Considering the fact that I haven`t seen the above mentioned solution in mathematically-methodological literature, I believe that the inovated version of the pithagorean solution of construction of the regular pentagon with a given side a, would be very similar to it .

Let us see how the young K.F.Gauss collected the first 100 integers. While the teacher is expected that students add up the order of the first hundred numbers on a mission to solve the following way [1].

$$1+2+3+\ \dots\ +100=1+2+3+\ \dots\ +50+51+\ \dots\ +98+99+100=$$
$$=(1+100)+(2+99)+(3+98)+\ \dots\ +(50+51)=50\cdot 101=5050.$$

Previous example we can use to start a conversation with students about other possibilities of solving the same task. Specifically, students should be asked whether the task can be solved in a more efficient manner. If students cannot discover new heuristics, they need to set up an auxiliary question: What would happen if the union of pairs of numbers to make the sum of 100? Then some of them could easily be another solution. In any case, we should insist on the Confucian wisdom to the students through their thinking and work and find other solutions.

Of course, this is not the end of the story, because

$$1+2+3+\dots+100=1+2+3+\dots+50+51+\dots+98+99+100=$$
$$=(1+99)+(2+98)+\dots+(49+51)+(50+100)=49\cdot 100+150=$$
$$=5050$$

the task can be solved in the following manner:

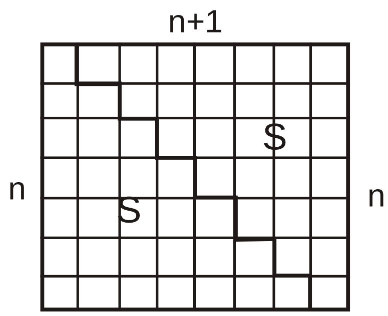
This typical example of motivation gives us

$$1+2+3+\ \dots\ +100=1+2+\ \dots\ +10+(10+1+10+2+\ \dots\ +10+9+10+10)+$$
$$+(20+1+20+2+\ \dots\ 20+10)+\ \dots\ +(90+1+90+2+\ \dots\ +90+10)=$$
$$=(1+2+\ \dots\ +10)\cdot 10+(1+2+\ \dots\ +9)\cdot 100=55\cdot 10+45\cdot 100=5050.$$

the opportunity to introduce students to the historical story of the results of the school of Pythagoras, the Greek mathematics. Pythagoreans emphasized that enhance math above commercial basis. This is evidenced by the following examples of their knowledge of arithmetic and the evidence that was used in geometry.

1. This is how they found out to what is the sum of the first n natural number s equal to

S = 1+2+3+...+n. Using the knowledge of the area of rectangle and picture 9. they stated that:

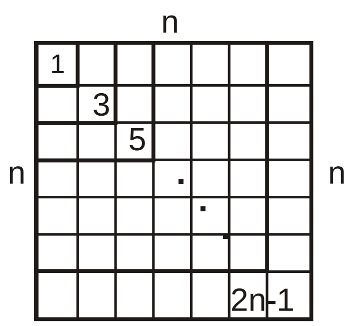


Picture 9.

2·S = n·(n+1), from where it follows that:

$$S=\frac{n\cdot(n+1)}{2}\quad [8].$$

2. In the similar way they obtained the formula for the sum of the first n odd natural numbers:



Picture 10.

X = 1+3+5+...+2n–1.

They simply noted that the sum of gnomons 1,3,5,...,2n–1 make the square picture 10. Then after calculating that are they stated that: $X = n^2$.

To the sum of the first n even numbers they came by using pure arithmetic and the result of the first problem.

$Y=2+4+6+...+2\cdot n=2\cdot(1+2+3+...+n)=2\cdot S=n\cdot(n+1)$.

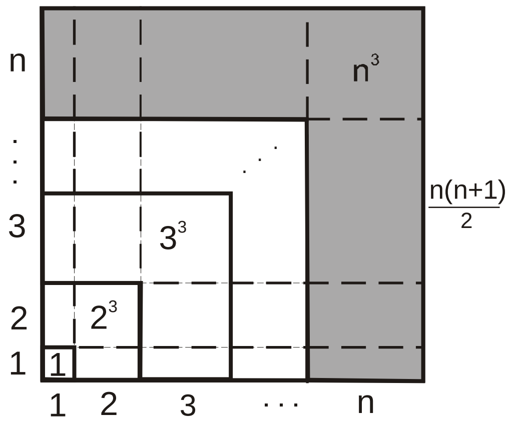
Using the previous result Pithagoreans found out in a different way that the sum of the first n odd numbers is equal to n^2.

$1+3+5+...+2n-1 = (2\cdot1-1)+(2\cdot2-1)+(2\cdot3-1)+...+ +2n-1=2\cdot(1+2+3+...+n)-n=n\cdot(n+1)-n=n^2$.

It is necessary to point out to students here, the historical development of the ideas, from these Pithagoreans solutions all the way to Blez Pascal`s recurrent formulas, observed through the prism of geometrical interpretations. The name of Nikomah from Geras can`t be avoided on that way (about 100. years). He groups the even numbers in the following way:

$$1+(3+5)+(7+9+11)+(13+15+17+19)+...+ \\ +(n\cdot(n+1)-2\cdot n+1+...+n\cdot(n+1)-1)=P$$

It is noted that the sum in every bracket is equal to the third degree of its members. It is easy to



Picture 11.

calculate that the number of these elements in brackets is:

$1+2+3+...+n=\dfrac{n(n+1)}{2}$, that is that there are that many even numbers in all brackets,so:

$$P=1+(3+5)+(7+9+11)+...=\left[\frac{n(n+1)}{2}\right]^2.$$

The same result was found by the Arabian mathematician Alkharhi in the 11. century. This is how he did that nice geometrical interpretation with the use of calculation of the sum of the area s of disjunctive gnomons.

$$P = 1^3 + 2^3 + 3^3 +...+ n^3$$

It is easy to see that the area of the n gnomon (marked in picture 11) is equal:

$2\cdot n\cdot\dfrac{n\cdot(n+1)}{2}-n^2=n^3$, so the surface of the square P is equal to the sum of the cubes of the

$$S_1=1^2+2^2+3^2+...+n^2=1\cdot2+2\cdot3+3\cdot4+...+n\cdot(n+1)- \\ -(1+2+3+...+n)=\frac{n\cdot(n+1)\cdot(2n+1)}{6}.$$

first n numbers:

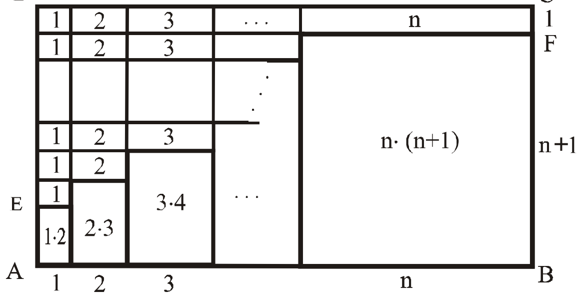
$$P=\left[\frac{n\cdot(n+1)}{2}\right]^2.$$

Here`s another example similar to the previous one, which is the achievement of the Arabic neopithagoreans from the 11th century.

Find: $S = 1\cdot2 + 2\cdot3 + 3\cdot4+...+ +n\cdot(n+1)$. This is the discovery they reached with the application of geometry and the calculation of surfaces. They previously transformed the above expression:

$$\frac{S}{2}=\frac{1\cdot2}{2}+\frac{2\cdot3}{2}+...+\frac{n(n+1)}{2}=1+(1+2)+ \\ +(1+2+3)+.....+(1+2+...+n).$$

Mark the surface of the rectangle ABCD with P (picture 12). It`s easy to see that:



Picture 12.

$$P = P_{AE..FBA} + P_{DE.....FCD} = 1\cdot 2 + 2\cdot 3 + ... + n\cdot(n+1) + 1 + (1+2) +$$
$$+ ... + (1+2+...+n) = S + \frac{S}{2}, \ \ and \ \ as:$$

$$P = \frac{n(n+1)}{2}\cdot(n+2) = \frac{n(n+1)(n+2)}{2} = \frac{3S}{2},$$
$$it \ \ is \ \ \ S = \frac{n(n+1)(n+2)}{3}.$$

Using the previous formula they also calculated the sum of the squares of the first n natural numbers: $1^2 + 2^2 + 3^2 + ... + n^2 = S_1$. With the following transforming and summing they found out: $1^2 = 1 \cdot (1+1) - 1 = 1 \cdot 2 - 1$

$2^2 = 2 \cdot (2+1) - 2 = 2 \cdot 3 - 2$

$3^2 = 3 \cdot (3+1) - 3 = 3 \cdot 4 - 3$

…………………………

$n^2 = n \cdot (n+1) - n = n\,(n+1) - n$

$$S_1 = 1^2 + 2^2 + 3^2 + ... + n^2 = 1\cdot 2 + 2\cdot 3 + 3\cdot 4 + ... + n\cdot(n+1) -$$
$$-(1+2+3+...+n) = \frac{n\cdot(n+1)\cdot(2n+1)}{6}\ .$$

Six centuries later Blez Paskal (1623 – 1662.g) as a seventeen years old, discovered the famous recurrent formula for the calculation of all the sums [1], [6]:

$$\begin{aligned} 1 + 2 + 3 + \ .....\ + n &= S_1 \\ 1^2 + 2^2 + 3^2 + \ .....\ + n^2 &= S_2 \\ 1^3 + 2^3 + 3^3 + \ .....\ + n^3 &= S_3 \\ ... + \ ... + \ ... + \ .....\ + \ ... &= ... \\ 1^m + 2^m + 3^m + \ .....\ + n^m &= S_m \\ m, n \in N \ \ , \end{aligned}$$

And he used the knowledge of his ingenious predecessors. The story about Pascal and the recurrent formulas is a story about algebra, but it is also evident that we can hardy separate what belongs to algebra and what to geometry, through the historical story about the quest for methods and heuristic in mathematics, particularly if we know that mathematics has had a geometry frame ever since Newtn. During practice and revision lessons in the II grade of the Gymnasium (Scientific department) and some high schools where mathematics is taught with the program of four lesson a week or on the lessons of the extra-curriculum teaching, the follow-up of this lecture could be completed with the presentation of Pascal`s formulas and thus giving the subject a combined, polyform shape:

$$\binom{m+1}{1}\sum_{k=1}^{n} k^m + \binom{m+1}{2}\sum_{k=1}^{n} k^{m-1} + ... + \binom{m+1}{m}\sum_{k=1}^{n} k = (n+1)^{m+1} -$$
$$-(n+1), \ \ m, n \in N.$$

Here is the methodological version of my reconstruction of that derivation: To determine the sum of the first n natural numbers S_1 Pascal uses the formula $q^2 + 2\cdot q + 1 = (q+1)^2$ taking successively $(q = 1,2,3,\ .....\ ,n)$ and getting the equations:

$$\begin{aligned} &for \ \ q = 1 && 1^2 + 2\cdot 1 + 1 = 2^2 \\ &for \ \ q = 2 && 2^2 + 2\cdot 2 + 1 = 3^2 \\ &.. \\ &.. \\ &for \ \ q = n-1 && (n-1)^2 + 2\cdot(n-1) + 1 = n^2 \\ &for \ \ q = n && n^2 + 2\cdot n + 1 = (n+1)^2, \end{aligned}$$

With whose addition he gets: $1 + 2\cdot\left[1 + 2 + 3 + \ .....\ + (n-1) + n\right] + n = (n+1)^2,$ that is. $2\cdot S_1 = (n+1)^2 - (n+1)$, which is equivalent to the expression $\binom{2}{1}\cdot S_1 = (n+1)^2 - (n+1)$, so $S_1 = \frac{n\cdot(n+1)}{2}$.

To determine the sum of the squares of the first n natural numbers S_2 he uses the formula $q^3 + 3\cdot q^2 + 3\cdot q + 1 = (q+1)^3$ taking successively for $(q = 1,2,3,\ .....\ ,n)$ and gets the equations :

$$\begin{aligned} &q = 1 && 1^3 + 3\cdot 1^2 + 3\cdot 1 + 1 = 2^3 \\ &q = 2 && 2^3 + 3\cdot 2^2 + 3\cdot 2 + 1 = 3^3 \\ &... \\ &... \\ &q = n-1 && (n-1)^3 + 3\cdot(n-1)^2 + 3\cdot(n-1) + \\ & && + 1 = n^3 \\ &q = n && n^3 + 3\cdot n^2 + 3\cdot n + 1 = (n+1)^3 \end{aligned}$$

By addition of the left and right sides of all of this *n* equation Pascal forms the following equation:

$$1^3+3\cdot\left[1^2+2^2+3^2+\ \ldots\ +(n-1)^2+n^2\right]+$$
$$+3\cdot\left[1+2+3+\ \ldots\ +(n-1)+n\right]+n=(n+1)^3,$$ that is. $3\cdot S_2+3\cdot S_1=(n+1)^3-(n+1)$, which is equivalent to the expression

$$\binom{3}{1}\cdot S_2+\binom{3}{2}\cdot S_1=(n+1)^3-(n+1),$$ so $$S_2=\frac{n\cdot(n+1)\cdot(2\cdot n+1)}{6}.$$

To determine the sum of cubes of the first n natural numbers S_3 the famous Frenchman used the formula $q^4+4\cdot q^3+6\cdot q^2+4\cdot q+1=(q+1)^4$ taking successively for $(q=1,2,3,\ \ldots\ ,n)$ and he got the equations:

$$\begin{array}{ll} \textit{for}\quad q=1 & 1^4+4\cdot1^3+6\cdot1^2+4\cdot1+1=2^4\\ \textit{for}\quad q=2 & 2^4+4\cdot2^3+6\cdot2^2+4\cdot2+1=3^4\\ \ldots\ldots & \\ \ldots\ldots & \\ \textit{for}\quad q=n-1 & (n-1)^4+4\cdot(n-1)^3+6\cdot(n-1)^2+\\ & +4\cdot(n-1)+1=n^4\\ \textit{for}\quad q=n & n^4+4\cdot n^3+6\cdot n^2+4\cdot n+1=(n+1)^4 \end{array}$$

By addition of the left and right sides of all of these n equations he gets:

$$1^4+4\cdot(1^3+2^3+\ \ldots\ +n^3)+6\cdot(1^2+2^2+\ \ldots\ +n^2)+$$
$$+4\cdot(1+2+\ \ldots\ +n)+n=(n+1)^4,$$

tj. $4\cdot S_3+6\cdot S_2+4\cdot S_1=(n+1)^4-(n+1)$, which is equivalent to the expression: $\binom{4}{1}S_3+\binom{4}{2}S_2+\binom{4}{3}S_1=(n+1)^4-(n+1)$, so when it is introduced into the previous equation for $S_1=\frac{n\cdot(n+1)}{2}$ i $S_2=\frac{n\cdot(n+1)\cdot(2\cdot n+1)}{6}$,

you get $S_3=\left[\frac{n\cdot(n+1)}{2}\right]^2$.

This procedure can be continued ant the you can reach the sum S_m using Pascal`s procedure that is binomial formula:

$$q^{m+1}+\binom{m+1}{1}\cdot q^m+\binom{m+1}{2}\cdot q^{m-1}+\ \ldots\ +\binom{m+1}{m}\cdot q+1=(q+1)^{m+1}$$

taking successively $(q=1,2,3,\ \ldots\ ,n)$ getting the equation:

$$\begin{array}{ll} \textit{for}\quad q=1 & 1^{m+1}+\binom{m+1}{1}\cdot1^m+\binom{m+1}{2}\cdot1^{m-1}+\ \ldots\ +\binom{m+1}{m}\cdot1+1=2^{m+1}\\ \textit{for}\quad q=2 & 2^{m+1}+\binom{m+1}{1}\cdot2^m+\binom{m+1}{2}\cdot2^{m-1}+\ \ldots\ +\binom{m+1}{m}\cdot2+1=3^{m+1}\\ \ldots\ldots & \\ \ldots\ldots & \\ \textit{for}\ q=n-1 & (n-1)^{m+1}+\binom{m+1}{1}\cdot(n-1)^m+\binom{m+1}{2}\cdot(n-1)^{m-1}+\ \ldots\ +\binom{m+1}{m}\cdot(n-1)+1=n^{m+1}\\ \textit{for}\quad q=n & n^{m+1}+\binom{m+1}{1}\cdot n^m+\binom{m+1}{2}\cdot n^{m-1}+\ \ldots\ +\binom{m+1}{m}\cdot n+1=(n+1)^{m+1} \end{array}$$

By addition of the left and right sides of all of these n equations Pascal forms the following equation:

$$1^{m+1}+\binom{m+1}{1}(1^m+2^m+\ldots+n^m)+\binom{m+1}{2}(1^{m-1}+2^{m-1}+\ldots+n^{m-1})+\ldots+\binom{m+1}{m}(1+2+\ldots+n)+n=(n+1)^{m+1}$$

$$\binom{m+1}{1}\cdot S_m+\binom{m+1}{2}\cdot S_{m-1}+\ \ldots\ +\binom{m+1}{m}\cdot S_1=(n+1)^{m+1}-(n+1)$$

which is equivalent to the expression:

$$\binom{m+1}{1}\cdot\sum_{k=1}^{n}k^m+\binom{m+1}{2}\cdot\sum_{k=1}^{n}k^{m-1}+\ \ldots\ +\binom{m+1}{m}\cdot\sum_{k=1}^{n}k=(n+1)^{m+1}-(n+1),\quad m,n\in N,$$

by which the heuristic of Pascal`s formula is finally reconstructed [8].

Combined polyform geometrical and non geometrical interpretations, given „in the light" of these or similar reconstructions and innovations, have positive influence on the understanding and cognition of different mathematical and many non-mathematical phenomena. Such application of the principle of polyform, given as an integral dialectic unity of all the variations of similarities or opposites, and even paradoxical non standard things, of teaching contents and their didactic transpositions, always produces dynamism and activation of a teaching process and implies additional motivational effects for students.

References

[1] Jan Struik J. Dirk, Summary history of mathematics, Institute for Textbooks and Teaching Aids, Belgrade, 1991.

[2] Lietzmann Waltert, Methodik des mathematischen Unterrichts IV, Heidelberg, 1955.

[3] Butler – Wren: The Teaching of Secondary Mathematics, The Mc Graw-Hill book company, New York, 1960.

[4] M. Marjanovic, M. Prvanovic, M. Perovic, R. Dacic, D. Trifunovic, V.A. Vujicic and Z. Mamuzic, Historical writings on mathematics And mechanics, Mathematical Institute, Belgrade, 1989.

[5] И.Ф. Шаригин, *Нужна ли школе 21-го века Геометрия?* Математическое просвещение, (Третья серия) Издательство МЦНИО, Москва. 2004. 34 – 52

[6] M. Perovic, Development of mathematical ideas and philosophical - script, specialist postgraduate studies, Faculty ofPodgorica, 2004.

[7] Dj.G. Markovic, New views on methods of teaching mathematics, Macarius, Podgorica, 2008.

[8] Marković, Đoko G. (2006). Geometrical polymorphism. (Geometrijski polimorfizam.) (Serbian) Nast. Mat. 51, No. 3-4, 11-24. www.zentralblatt-math.org/**matheduc**/en/?q=an%3A2008b.00197

Coresponding autor: *Djoko G. Markovic*
Institution: *Faculty of philosophy in Niksic, Montenegro*
E-mail: *djokogm@hotmail.com*