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One Approach for Solving Trigonometric Equations Using Complex in the Mathematical Education

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***Abstract* – The goal of this development is introducing a reader the solution of one class comprising trigonometric equations in the Teaching Course of Mathematics by using trigonometric form of the complex numbers. An exemplary approach to solving these equations, suitable for students from 11th to 12th grade, as well as for pupils participating in mathematical camps, olympiads, mathematical competitions, computer mathematics olympiads is considered.**

***Keywords* – trigonometric equations, talented pupils, complex number, complex analysis, function.**

1. Introduction

Intensive development of the technique, information and communication technologies enforces an impression on formation and development the interest for learning among the grownup. On the other hand, in contemporary conditions in which mathematicians involved in science and technique arose the necessity of continuous looking for and implementation in different areas of well qualified specialists of mathematics with interest and capacity for creative activity.

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These things request to enforce necessity of creation, education and making of talented pupils with outstanding mathematical accomplishment and interest for creative development. In is maybe the cases in which there is a concern that mathematics and motivation for studying are provoked from the practical application of the mathematical knowledge and their relations with other spheres of the science.

Apart from the mathematics is the trigonometry, which is in close cooperation with the ancient development of astronomy, surveying ships, building, cartography and other fields of the social life.

On the contemporary stage in the class and outclass work of mathematics in the Bulgarian secondary schools the trigonometric functions $\sin x, \cos x, \mathrm{tg} x, \cot g x$, trigonometric equations and in equations are very important. In the teaching and in methodological literature the key competences are shown, which are mandatory for the pupils in secondary schools:

- know the basic trigonometric equations and to solve the equations switch – it can be reduced to them;
- can find values of a parameter given in advance for certain conditions comprising trigonometric equations and inequalities, inequalities that can be reduced to rational ones;
- can take the properties of trigonometric functions [8], [11];
- acquire trigonometric formulas [3], [4], [11];
- have a background for complex numbers and the correspondence with the points of the plane;
- know the algebraic and trigonometric forms of the complex numbers and notion related to them, as well as, know how to use them;
- know the operations with the complex numbers and apply them [7], [9], [14];
- know the Moivre formulas and find zeroes of polynomials with real coefficients [7], [9], [14].

The expected results are as follows: They are able to solve trigonometric equations and inequalities by using of different methods; to perform operation with complex number.

In the trigonometry exists a great variety of trigonometric equations, that require different methods for solving them. However, in the school course only a few are considered – trigonometric equations and methods for their solving. Only equations, that could be reduced to the basic trigonometric equations: $\sin x = a$, $\cos x = a$, $tgx = a$, $\cot gx = a$ are considerd. Also, equations which contain only one function with the same argument are considered as well. Then the solving is connected with substitution, which leads to application of knowledge for algebraic equations solving. If the algebraic equation has a solution, then it is reduced again to solving the basic trigonometric equations. But if the equation contains trigonometric functions with a common argument, then again it is suitable the substitution method to reduce the original equation to equation with only one unknown $tg\frac{x}{2}$.

For studying the trigonometric knowledge the investigation and transformation of different trigonometric expressions for the solution of the trigonometric equations are very important [12].

It is useful to consider alternative approaches [2], to solve equations and prove the issues. It is interesting for the pupils with outstanding mathematical capibilities the equations of the form $a\cos x + b\sin x = c$. The methodological directives for solution of such kind of equations are connected with decomposition of mulipliers (the equations of this kind [8] one solves by decomposition, as all addends one transfers at one hand-side and the resulting expression is presented as a product) or by application of the formula $\sin(\alpha \pm \beta)$ or $\cos(\alpha \pm \beta)$, for this one must divide the both sides of the equation with $\sqrt{a^2 + b^2}$ [10]. In [5] the equation is divided by $\sqrt{a^2 + b^2}$ and a complimentary angle ψ is added.

Some pupils have difficulties with solving this kind of equations, because it is a problem to find the expression for multiplication or division of the both sides of the equation. In the present development we propose a new approach, that activity facilitates, in which pupils for solving certain substitution prepare concrete formulas in advance deduced from the teacher.

In Bulgaria this approach did not apply up to now in the school course of mathematics. It is not known even, in other countries' publications, in the teaching and methodology literature for application of this approach to solve this class of equations. In fact the problems, concerning solution of these equations are suitable for pupils of 11^{th} - 12^{th} grades form, moreover for pupils who participate in math campus, Olympiads, mathematical completions, Olympiads within computer mathematics field.

The implementation of this approach for solution of trigonometric equations aims to give pupils additional mathematical knowledge, to prepare one directional preparation for studying elements of the mathematical analysis in the next stages of education. On the base of this good training the pupils will take interest in mathematics and its applications, they will get a fundament for successful continuation of their education at the universities.

2. Exposition the Method

This section considers conditions necessary for the existence of solutions for equation, as well as methods for their finding on the case of Euler's formula.

Theorem 1 [5]. Let a, b, c are real numbers and let the inequality $a^2 + b^2 \geq c^2$, holds then the trigonometric equations

$$a\cos x + b\sin x = c \qquad (1)$$

have real solutions.

Proof: We multiply the two sides of equations (1) by $\frac{1}{\sqrt{a^2+b^2}}$ to consider the equality

$$\frac{a}{\sqrt{a^2+b^2}}\cos x + \frac{b}{\sqrt{a^2+b^2}}\sin x = \frac{c}{\sqrt{a^2+b^2}}.$$

The equality

$$\left(\frac{a}{\sqrt{a^2+b^2}}\right)^2 + \left(\frac{b}{\sqrt{a^2+b^2}}\right)^2 = 1$$

implies, the existence of angle ψ, such that

$$\sin\psi = \frac{a}{\sqrt{a^2+b^2}} \text{ and } \cos\psi = \frac{b}{\sqrt{a^2+b^2}}.$$

In this case we have

$$\sin\psi\cos x + \cos\psi\sin x = \frac{c}{\sqrt{a^2+b^2}}$$

Then, there exists a number x, such that

$$\sin(\psi + x) = \frac{c}{\sqrt{a^2+b^2}},$$

i.e. the equation (1) has a solution.

Further, we use the Euler's formula [6]:

$$e^{ix} = \cos x + i \sin x, \qquad (2)$$

to prove the Theorem 2.

Theorem 2. If $a^2 + b^2 \geq c^2$, then the solutions of equation (1) are as follows:

$$x_1 = u_1 \arccos\left(\frac{ac - b\sqrt{a^2 + b^2 - c^2}}{a^2 + b^2}\right) + 2k\pi;$$

$$x_2 = u_2 \arccos\left(\frac{ac + b\sqrt{a^2 + b^2 - c^2}}{a^2 + b^2}\right) + 2k\pi,$$

in which

$$u_1 = \operatorname{sign}\left(\frac{bc + a\sqrt{a^2 + b^2 - c^2}}{a^2 + b^2}\right);$$

$$u_2 = \operatorname{sign}\left(\frac{bc - a\sqrt{a^2 + b^2 - c^2}}{a^2 + b^2}\right).$$

Proof: By the change of x with $-x$ in (2) and summing by terms and subtraction of the results, we get $\sin x$ and $\cos x$:

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \ \sin x = \frac{e^{ix} - e^{-ix}}{2i}. \qquad (3)$$

We substitute (3) in (1):

$$a\frac{e^{ix} + e^{-ix}}{2} + b\frac{e^{ix} - e^{-ix}}{2i} = c \quad /.e^{ix}$$

$$\frac{a}{2}\left(\left(e^{ix}\right)^2 + 1\right) + \frac{b}{2i}\left(\left(e^{ix}\right)^2 - 1\right) = ce^{ix}$$

We set $e^{ix} = t$

$$\frac{a}{2}\left(t^2 + 1\right) + \frac{b}{2i}\left(t^2 - 1\right) = ct$$

$$\frac{a - ib}{2}t^2 - ct + \frac{a + ib}{2} = 0$$

$$D = c^2 - 4.\frac{1}{2}(a - ib)\frac{1}{2}(a + ib) = c^2 - \left(a^2 + b^2\right)$$

$$t_{1/2} = \frac{c \pm \sqrt{c^2 - \left(a^2 + b^2\right)}}{a - ib}.\frac{a + ib}{a + ib}$$

$$= \frac{a}{a^2 + b^2}\left(c \pm \sqrt{c^2 - \left(a^2 + b^2\right)}\right) +$$

$$+ i\frac{b}{a^2 + b^2}\left(c \pm \sqrt{c^2 - \left(a^2 + b^2\right)}\right)$$

It follows from Theorem 1, that in order equation (1) to have real solutions

$$c^2 - \left(a^2 + b^2\right) \leq 0 \Rightarrow \sqrt{c^2 - \left(a^2 + b^2\right)} = i\sqrt{a^2 + b^2 - c^2}.$$

$$t_{1/2} = \frac{a}{a^2 + b^2}\left(c \pm i\sqrt{a^2 + b^2 - c^2}\right) +$$

$$+ i\frac{b}{a^2 + b^2}\left(c \pm i\sqrt{a^2 + b^2 - c^2}\right) \qquad (4)$$

$$= \underbrace{\frac{ac \mp b\sqrt{a^2 + b^2 - c^2}}{a^2 + b^2}}_{A} + \underbrace{\frac{bc \pm a\sqrt{a^2 + b^2 - c^2}}{a^2 + b^2}}_{B}i$$

The signs $\mp$ and $\pm$ must be taken in correspondent sequence. One takes the positive value of the root.

By letting $t = e^{ix}$, one obtains for (2) and (4)

$$t = e^{ix} = \cos x + i \sin x$$

$$= \frac{ac \mp b\sqrt{a^2 + b^2 - c^2}}{a^2 + b^2} + \frac{bc \pm a\sqrt{a^2 + b^2 - c^2}}{a^2 + b^2}i$$

$$= A + Bi.$$

One equalizes the real and imagined parts to find:

$$\cos x = \frac{ac \mp b\sqrt{a^2 + b^2 - c^2}}{a^2 + b^2} \qquad (5.1)$$

$$\sin x = \frac{bc \pm a\sqrt{a^2 + b^2 - c^2}}{a^2 + b^2} \qquad (5.2)$$

It is easy to be seen, that $A^2 + B^2 = 1$.

The solutions of (1) are given by the systems:

$$\left|\begin{array}{l} \cos x = \underbrace{\dfrac{ac - b\sqrt{a^2 + b^2 - c^2}}{a^2 + b^2}}_{A_1} \\ \sin x = \underbrace{\dfrac{bc + a\sqrt{a^2 + b^2 - c^2}}{a^2 + b^2}}_{B_1} \end{array}\right. \cup \left|\begin{array}{l} \cos x = \underbrace{\dfrac{ac + b\sqrt{a^2 + b^2 - c^2}}{a^2 + b^2}}_{A_2} \\ \sin x = \underbrace{\dfrac{bc - a\sqrt{a^2 + b^2 - c^2}}{a^2 + b^2}}_{B_2} \end{array}\right. \qquad (6)$$

Therefore, the solutions of (1) are:

$$\begin{aligned} x_1 &= \operatorname{sign}(B_1)\arccos(A_1) + 2k\pi; \\ x_2 &= \operatorname{sign}(B_2)\arccos(A_1) + 2k\pi. \end{aligned} \qquad (7)$$

This deducing is a training of the algebra on complex numbers. It can be done with pupils of the last classes and as corollary the formulas for direct solving of equation (1). If they are not taught profoundly, the teacher gives prepared formulas (6) and (7).

Using elements of the higher mathematics is not always a good idea. For this, a compromise must be reached between the comprehensibility of a given situation presented at a lower level and their ability to participate in a broader sphere at a higher level [1].

3. Application

In this section the applications of the theorems proved are illustrated. In the first problem the equation has two solutions. At second problem one obtains one, root and the third one the solutions are connected by a parameter. In some parametric equations, apart from the traditional approach to solution, it is appropriate to give a graphical approach to the solution [13].

Problem 1. Solve the equation $\sqrt{3}\cos x + 3\sin x = 3$.

Solution:

I-st approach: Write the equation in the form

$$\frac{\sqrt{3}}{2\sqrt{3}}\cos x + \frac{3}{2\sqrt{3}}\sin x = \frac{3}{2\sqrt{3}} \Rightarrow \frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x = \frac{\sqrt{3}}{2}$$

Since $\sin\frac{\pi}{6} = \frac{1}{2}$ and $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, the expression $\frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x$ can be written as

$$\sin\frac{\pi}{6}\cos x + \cos\frac{\pi}{6}\sin x = \sin\left(\frac{\pi}{6} + x\right).$$

Therefore, the equation is equivalent to the equation $\sin\left(\frac{\pi}{6} + x\right) = \frac{\sqrt{3}}{2}$. The solutions of the last one are as follows:

$$\frac{\pi}{6} + x = \frac{\pi}{3} + 2k\pi \Rightarrow x = \frac{\pi}{6} + 2k\pi,\ \ k = 0,\ \pm 1,\ \pm 2, \ldots$$

$$\frac{\pi}{6} + x = \pi - \frac{\pi}{3} + 2k\pi \Rightarrow x = \frac{\pi}{2} + 2k\pi,$$
$$k = 0,\ \pm 1,\ \pm 2, \ldots$$

II approach: According Theorem 1 [5] in order the equation $\underbrace{\sqrt{3}}_{a}\cos x + \underbrace{3}_{b}\sin x = \underbrace{3}_{c}$, to have real roots, the inequality $a^2 + b^2 - c^2 \geq 0$. Therefore $3 + 9 - 9 \geq 0$ is fulfilled.
Taking into account (6), we write down, we note:

$$\left|\begin{array}{l} \cos x = \dfrac{3\sqrt{3} - 3\sqrt{3}}{12} \\ \sin x = \dfrac{9+3}{12} \end{array}\right. \cup \left|\begin{array}{l} \cos x = \dfrac{3\sqrt{3} + 3\sqrt{3}}{12} \\ \sin x = \dfrac{9-3}{12} \end{array}\right.$$

$$\left|\begin{array}{l} \cos x = 0 \\ \sin x = 1 \end{array}\right. \cup \left|\begin{array}{l} \cos x = \dfrac{\sqrt{3}}{2} \\ \sin x = \dfrac{1}{2} \end{array}\right.$$

Therefore, the solution of the equations are:

$$x_1 = \operatorname{sign}(1)\arccos(0) + 2k\pi = \frac{\pi}{2} + 2k\pi, k \in Z;$$

$$x_2 = \operatorname{sign}\left(\frac{1}{2}\right)\arccos\left(\frac{\sqrt{3}}{2}\right) + 2k\pi = \frac{\pi}{6} + 2k\pi, k \in Z.$$

Problem 2. Solve the equation $\cos x + \sin x = \sqrt{2}$.

Solution:

I approach: One rewrite the equation in the form $\frac{\sqrt{2}}{2}\cos x + \frac{\sqrt{2}}{2}\sin x = 1$. Since $\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ and $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, the expression $\frac{\sqrt{2}}{2}\cos x + \frac{\sqrt{2}}{2}\sin x$ can be written as $\sin\frac{\pi}{4}\cos x + \cos\frac{\pi}{4}\sin x = \sin\left(\frac{\pi}{4} + x\right)$.

Therefore the given equation is equivalent to the equation $\sin\left(\frac{\pi}{4} + x\right) = 1$. The solutions of the last equation are as follows:

$$x_1 = x_2 = \frac{\pi}{4} + 2k\pi,\ \ k = 0,\ \pm 1,\ \pm 2, \ldots$$

II approach: It follows from Theorem 1

$a^2 + b^2 - c^2 \geq 0$. Let $a = 1$, $b = 1$ and $c = \sqrt{2}$. In view of (6), we have:

$$\left|\begin{array}{l}\cos x = \dfrac{\sqrt{2}}{2} \\ \sin x = \dfrac{\sqrt{2}}{2}\end{array}\right. \cup \left|\begin{array}{l}\cos x = \dfrac{\sqrt{2}}{2} \\ \sin x = \dfrac{\sqrt{2}}{2}\end{array}\right.$$

Therefore, the solutions of the equation are:

$$x_1 = x_2 = \mathrm{sign}\left(\frac{\sqrt{2}}{2}\right)\arccos\left(\frac{\sqrt{2}}{2}\right) + 2k\pi = \frac{\pi}{4} + 2k\pi,$$

$k \in Z.$

Problem 3. Solve the equation
$3\cos x \sin t - \sin x \cos t - 4\cos t = 3\sqrt{3}$.

Solution:

By a, b and c we denote $3\sin t$, $-\cos t$ and $3\sqrt{3} + 4\cos t$, respectively.

If the inequality holds $a^2 + b^2 - c^2 \geq 0$, then Theorem 1 implies that the equation has solution. From the assumption the inequalities hold:

$$9\sin^2 t + \cos^2 t - 27 - 24\sqrt{3}\cos t - 16\cos^2 t \geq 0;$$

$$9\sin^2 t + 9\cos^2 t - 9\cos^2 t - 27 - 24\sqrt{3}\cos t - 15\cos^2 t \geq 0;$$

$$4\cos^2 t + 4\sqrt{3}\cos t + 3 \leq 0;$$

$$\left(2\cos t + \sqrt{3}\right)^2 \leq 0.$$

It follows from here, that $2\cos t + \sqrt{3} = 0$, $\cos t = -\dfrac{\sqrt{3}}{2}$ and therefore, for $t = \pm\dfrac{5\pi}{6} + 2k\pi$, then $a^2 + b^2 - c^2 = 0$.

One can check directly that for these values of t the inequalities $a^2 + b^2 - c^2 \geq 0$ holds.

We consider two cases:

Case 1. For $t = -\dfrac{5\pi}{6} + 2k\pi$ we have

$$a = 3\sin\left(-\frac{5\pi}{6}\right) = -\frac{3}{2};$$

$$b = \frac{\sqrt{3}}{2};$$

$$c = 3\sqrt{3} - 4.\frac{\sqrt{3}}{2} = \sqrt{3};$$

$$\left|\begin{array}{l}\cos x = \dfrac{ac}{a^2 + b^2} = -\dfrac{\sqrt{3}}{2} \\ \sin x = \dfrac{bc}{a^2 + b^2} = \dfrac{1}{2}\end{array}\right.;$$

$$x_1 = \mathrm{sign}\left(\frac{1}{2}\right)\arccos\left(-\frac{\sqrt{3}}{2}\right) + 2k\pi = \frac{5\pi}{6} + 2k\pi,$$

$k \in Z.$

Case 2. For $t = \dfrac{5\pi}{6} + 2k\pi$ then, we have

$$a = \frac{3}{2};\ b = \frac{\sqrt{3}}{2};\ c = \sqrt{3};$$

$$\left|\begin{array}{l}\cos x = \dfrac{\sqrt{3}}{2} \\ \sin x = \dfrac{1}{2}\end{array}\right.;$$

$$x_2 = \mathrm{sign}\left(\frac{1}{2}\right)\arccos\left(\frac{\sqrt{3}}{2}\right) + 2k\pi = \frac{\pi}{6} + 2k\pi,$$

$k \in Z.$

The approach described permits that understanding can be easier and better about different activities for solution for a class dealing with trigonometric equations, which usually one solves in advance by given mechanical prescription. In many cases, when a prescription is set in advance, one performs mechanically, without understanding. The better and easier to understand the logic of the solution requires becoming the object of the successive teaching regarding the nature of notation of trigonometric equation, and the methods of its solving.

As in many other fields, the teachers do not follow immediately the mathematical discipline through the books. This could be explained, and one can say that it is „hygienically". On the other hand, the reset to a new approach, method, language or technology requires skills and time for its assimilation is needed.

The present approach for solving the considered type equations is not used in the teaching practice, but is connected with important knowledge in the mathematics.

4. Conclusion

The present approach is addressed to pupils, participating mathematical camps, as well to pupils who attend mathematical schools and for pupils' preparation for Olympiads, competitions and future mathematics students. It will be useful for teachers, who wish to deepen and extend their knowledge and skills to practice solving trigonometric equations.

The present development will be of interest for alls, who wish to enrich and to get acquainted with the methods and processes of solving the trigonometric equations, in order to increase their individual mathematical culture.

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