Diophantine Geometry in Space E_2 and E_3

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Abstract – **In this paper, we are focusing on solving Diophantine equations with geometry. Integers that are solutions of Diophantine equations are represented in space by a lattice. We will use the fact that equations can be interpreted geometrically as curves or surfaces. To determine the properties of the solutions, we will use tangents (of the surface).**

Keywords **– Geometry, Equations, Space, Surface, Solutions.**

1. Introduction

In this paper, we are focusing on using geometry to solve Diophantine equations. The use of geometry in number theory tasks was among the primary solutions. The reason for this was that the geometric view often offered a clear view of the problem solved, and the precision of geometric methods acted as an intuitive and logical procedure. Many processes and types of problem solving from number theory via geometry are written in the Euclidean work "Fundamentals". However, many tasks turned out to be difficult to solve using geometry, which prompted the emergence of a new division of mathematics called "Algebra". This division of mathematics has evolved independently and has brought about a number of discoveries in equation solutions and in the research of mathematical structures that have influenced the development of other parts of mathematics. The fundamental link between algebra and geometry occurred in the work "Geometry", written by René Descartes. In the work, Descartes introduces the coordinate system and thus builds the basics of analytical geometry. In addition to this, the mathematical field of "number theory" became independent. The Diophantine equations had a major influence on this area. However, the Diophantine equations were not distant from geometry, because many special problems in geometry were directed towards the synthesis of the Diophantine equation, and so these mathematical domains are interconnected. It was from this connection that work was being done to unify Diophantine equations and geometry. The connection between geometry and Diophantine equations was gradually crystallized with the introduction of a discrete space. The discrete space determined the basic properties of the solutions of Diophantine equations and thus provided the basic conditions for the solution sought. This resulted in one of the most elaborate works in the field of theory of Diophantine geometry, from the hands of Hermann Minkowski. In his work, he used lattice points, which represented whole numbers, to move to a discrete space. He characterizes the lattice L as a discrete topology inserted into the Euclidean space. Here we apply the idea of the lattice to the Diophantine equation in the form of $f(x,y)=c$ in the space E_2, so the Diophantine equation is represented here as a curve. Similarly, we can apply the idea to the equation in the form of $f(x,y,z)=c$ for E_3, which can be reformulated to the equation using the assumed solution in the form of $f(x,y)=d$ for E_2. More about Minkowski's number geometry theory can be found in the reference section [1]. We use the lattice topology in the article, determining the properties of the solution for the Diophantine equation [2], [3], [4], [5] by means of lattice points and a tangent. We apply this method using a tangent and lattice points to familiar and demanding tasks, such as Fermat's theorem and Beal's conjecture.

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2. Lattice points

In geometry, a Diophantine equation can be interpreted as a curve or surface. For surface equations, we determine the projection into the plane $x_i x_j$. Consequently, we can easily determine the lattice points in the space E_2, by specifying the method for the tangent from the textbook [6].
We will use the coordinate system $\langle O, x, y \rangle$ by specifying the integers by means of a scale on the axes x, y. Then, every point that is important to us has integer coordinates $A[x_0, y_0]$, $x_0, y_0 \in Z$, and therefore the coordinates are also lattice points. Next, we can use a line to connect the origin of the coordinate system $O[0,0]$ and $A[x_0, y_0]$.
For the line passing through the origin of the coordinate system, we can write the equation in the form of

$$y = \frac{y_0}{x_0} \cdot x$$

Let $(x_0, y_0) = 1$, then $\frac{y_0}{x_0} = q$

$$y = q \cdot x$$

For the tangent we can find the points which intersect the axes x, y. To determine such a line, we will use affine transformation.

Let

$$F(x, y) = \mathrm{c}$$

We can use the following equation for the tangent

$$(O - A) \cdot \mathrm{grad}(F[A]) = b$$

Another condition is

$$s \cdot \mathrm{grad}(F[A]) = 0$$

By joining the above equations we get

$$-s \cdot \mathrm{grad}(F[A]) + (A - O) \cdot \mathrm{grad}(F[A]) = b$$

Then we get

$$\big(s + (A - O)\big) \cdot \mathrm{grad}(F[A]) = b$$

We will use $s = \big(S_j - A\big)$, $j = x, y$ and an inverse vector $\mathrm{grad}(F[A])^{-1}$ in the space E_2

$$-\big(A - S_j + O - A\big) \cdot \mathrm{grad}(F[A]) \cdot \mathrm{grad}(F[A])^{-1} = b \cdot \mathrm{grad}(F[A])^{-1}$$

$$-\big(A - S_j + O - A\big) \cdot 1 = b \cdot \mathrm{grad}(F[A])^{-1}$$

Then we will use

$$S_x[s_x, 0]\ ,\ 1_x = (1,0)$$

$$(O - S_x) \cdot 1_x = b \cdot \mathrm{grad}(F[A])^{-1} \cdot 1_x$$

$$-s_x = b \cdot 1_x \cdot \mathrm{grad}(F[A])^{-1}$$

For the inverse vector is valid

$$\mathrm{grad}(F[A])^{-1} = \frac{\mathrm{grad}(F[A])^T}{\sqrt{\mathrm{grad}(F[A]) \cdot \mathrm{grad}(F[A])}}$$

For Diophantine equations, it is useful to know that $s_x \in Z$, while this is not a necessary condition. Other useful information is

$$(O - A) \cdot \mathrm{grad}(F[A])^{-1} = \alpha\ ;\ \alpha \in Q$$

By way of the tangent above the lattice, we can divide the Diophantine equation into the sets that determine the conditions for the solution above the set of integers. Furthermore, if we assume that there is a solution, the equation can be parameterized so as to solve tasks in E_3 and similarly in E_2 by creating a surface cut in the level z_0 to obtain the curve $f(x, y) = c$.

Then, we can determine the relationship between the tangent in the assumed solution and the line that originated from the beginning of the coordinate system with the expected solution:

$$\frac{x - x_0}{y - y_0} = \frac{x_0}{y_0} \cdot \alpha \qquad \alpha \in Q^+$$

2.1 Fermat's last theorem

One of the well-known problems in mathematics is Fermat's last theorem [7], [8], [9]. This is aimed at finding a solution to the Diophantine equation in the form of $x^n + y^n = z^n, n > 2$. We were motivated by the well-known Pythagorean triads in the form of $3^2 + 4^2 = 5^2$. Fermat pointed out that there are no integer solutions for the equation in the form of $x^3 + y^3 = z^3$ and also said that there is no solution between integers for any equation in the form of $x^n + y^n = z^n, n > 2$. Despite the convincing claim that he found evidence, no evidence was found. After nearly 300 years, it was the British mathematician Andrew Wiles who was able to prove this claim in 1994, improving upon what Fermat did not know.
Nonetheless, we will use the theory of lattice points for Fermat's last theorem. Let us consider the Diophantine equation in the form of

$$x^n + y^n = z^n\ , n > 2$$

$$\mathrm{grad}(F[A]) = (x_0^{n-1}, y_0^{n-1}, -z_0^{n-1})$$
$$(O - X) \cdot \mathrm{grad}(F[A]) = 0$$

Let us assume the existence of the solution and therefore determine the conditions in the way that we fix one coordinate and, for the remaining coordinates, determine the conditions resulting from the projection into the planes xy and xz

$$\frac{y - y_0}{x - x_0} = \frac{y_0^{n-1}}{x_0^{n-1}}$$

$$\frac{z - z_0}{x - x_0} = -\frac{z_0^{n-1}}{x_0^{n-1}}$$

Let us use the projection in the plane xy and fix z_0

$$\frac{y - y_0}{x - x_0} = \frac{y_0^{n-1}}{x_0^{n-1}}$$

Let $\alpha, \beta \in Q$

$$y - y_0 = \alpha y_0$$
$$x - x_0 = \beta x_0$$
$$\frac{\alpha}{\beta} = \frac{y_0^{n-1}}{x_0^{n-1}}$$

$$\alpha^{\frac{1}{n-1}} = y_0$$
$$\beta^{\frac{1}{n-1}} = x_0$$

$$\frac{\alpha y_0}{\beta x_0} = \frac{y_0^n}{x_0^n}$$
$$(x_0, y_0) = 1$$

Then we get $\alpha^{\frac{1}{n-1}} = y_0$, from which we assume that $\alpha \in N$ therefore $n = 2$ and $\alpha = y_0, \beta = x_0$.

2.2 Beal's conjecture

Just like Fermat's last theorem was a generalization of the Pythagorean triad, Beal's conjectureis a generalization of Fermat's last theorem [8]. If we change the exponent in the equation $x^n + y^n = z^n$ by replacing n with exponents a, b, c we get two interesting conjectures. The first one is Fermat-Catalan's conjecture. Fermat-Catalan's conjecture is focused on integer solutions of the equation in the form of:

$$x^a \pm y^b = z^c \, \frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$$

The conjecture says that all solutions of the equation fulfill the condition $(x, y, z) = 1$

The known solutions are:

$$1^a + 2^3 = 3^2$$
$$2^5 + 7^2 = 3^4$$
$$13^2 + 7^3 = 2^9$$
$$2^7 + 17^3 = 71^2$$
$$3^5 + 11^4 = 122^2$$
$$33^8 + 1549034^2 = 15613^3$$
$$1414^3 + 2213459^2 = 65^7$$
$$9262^3 + 15312283^2 = 113^7$$
$$17^7 + 76271^3 = 21063928^2$$
$$43^8 + 96222^3 = 30042907^2$$

Fermat-Catalan's conjecture says that the equation $x^a \pm y^b = z^c$ with the condition $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$ has a final number of solutions for a, b, c, x, y, z. The equation solutions are known in the publication [10].

On the other hand, Beal's conjecture uses the equation:

$$x^a + y^b = z^c \quad a, b, c < 2$$

This conjecture is intended to address the solutions with the condition $(x_0, y_0, z_0) = \delta$, $\delta \epsilon N$, for $a, b, c > 2$. This conjecture is linked by the basic equation to Fermat-Catalan's conjecture. Both of these conjectures are divided by the option, in which at least one of the exponents gains the value 2. On the other hand, if Fermat-Catalan's conjecture is true for $a, b, c > 2$, then Beal's conjecture is incorrect. For this reason, we will focus on Fermat-Catalan's conjecture and its necessity of an exponent of value 2.

We will begin with the assumption that we know at least one solution, for example $\mathrm{z}_0^{\mathrm{c}}$, $\mathrm{z}_0 > 0, \mathrm{c} > 2$. Based on this, we have a curve in the space E_2 to which we can determine a tangent line.

$$\mathrm{grad}(F[A]) = (a x_0^{a-1}, b y_0^{b-1})$$

$$(O - X) \cdot \mathrm{grad}(F[A]) = a x_0^a + b y_0^b$$

Using the tangent, we determine the condition:

$$\frac{y - y_0}{x - x_0} = \frac{b y_0^{b-1}}{a x_0^{a-1}}$$

Let $(x_0, y_0) = 1$ then we can determine:

$$\frac{x_0^{a-1}}{y_0^{b-1}} = \frac{\alpha}{\beta}, \qquad CGD(\alpha, \beta) = 1$$

Subsequently, $a < b$

$$\frac{x_0^{a-1}}{y_0^{a-1}} = \frac{\alpha}{\beta} y_0^{b-a}$$

Because $(x_0, y_0) = 1$ we can use:

$$\alpha \sim x_0^{a-1}, \qquad \beta \sim y_0$$

$$\beta_1 = \frac{\beta}{y_0^{b-a}}$$

We adjust:

$$\frac{x_0^{a-1}}{y_0^{a-1}} = \frac{\alpha}{\beta_1}$$

For the solutions sought, we can determine the conditions

$$x_0 = \alpha^{\frac{1}{a-1}}, \qquad y_0 = \beta_1^{\frac{1}{a-1}}$$

We will apply this on the equation

$$x^a + y^b = z^c$$

with the conditions for the solutions

$$(\alpha, \beta_1, z_0) = 1$$

$$\alpha^{\frac{a}{a-1}} + \beta_1^{\frac{b}{a-1}} = z^c$$

There are two options. The first option is $\alpha = x_0$, in which case the condition is met for $a = 2$. The second option is $\alpha = kx_0, k \in Q$.
We will then determine

$$k^a x_0^{\frac{a}{a-1}} = x_0^a$$

$$k = x_0^{\frac{a-2}{a-1}}$$

We will get a condition for $k \in Q$ only for $a = 2, k = 1$

3. Conclusion

In the paper, we have aimed to use the number geometry method (Minkowski), focusing on lattice points. The difference between classic Diophantine geometry and our solution is that we used a tangent and a defined lattice point by the tangent. Next, we applied this method to Fermat's last theorem and Beal's conjecture. We have shown that geometry in E_2 is a suitable instrument to find a character for solutions. We can see that the method using analytic geometry is a very fast method to find a character for the solution with a tangent and a lattice. The disadvantage is that we could not find the solution but only a character for the solution.

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