Lotka-Volterra Model with Two Predators and Their Prey

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Abstract **– In this paper will be observed the population dynamics of a three-species Lotka-Volterra model: two predators and their prey. This simplified model yields a more complicated dynamical system than classic Lotka-Volterra model. We will give the conditions under which one of the predators becomes extinct and when the coexistence between predators is possible. Given will be sufficient conditions for the existence of solutions for certain classes of Cauchy's solutions of Lotka-Volterra model. The behavior of integral curves in the neighborhoods of an arbitrary integral curves will be considered.**

Keywords **– Equilibrium Point, Growth Rate, Lotka-Volterra,Predator-Prey, Stability.**

1. Introduction

In the 1920s, the Italian mathematician Vito Volterra proposed a differential equation model to describe the population dynamics of two interacting species, a predator and its prey. He hoped to explain the increasing in predator fish (and so,decreasing in prey fish) in the Adriatic Sea during World War I.

There are several Lotka-Volterra predator prey models. The classic two dimensional Volterra-Lotka system

$$\begin{cases} \dot{x} = ax - bxy \\ \dot{y} = cxy - dy, \end{cases} \tag{1}$$

where the function $x(t)$ represents the populations of prey at time t, and also the function $y(t)$ represents the populations of predator at time t. All of the parameters a,b,c,d are nonnegative constants. The parameter a represents the natural growth rate of the prey in the absence of a predator. The parameter b represents the effect of predator in the prey population. Moreover if b is the only decreasing factor for the prey population, then prey will be eaten by predators. The parameter c represents the effect of prey in the predator population, moreover if c is the only increasing factor for the predator population, then the population growth is proportional to the food available. The parameter d represents the natural death rate of the predator in the absence of a prey.

If the prey population is large, the predators will have more food to support a larger population. However, when the predator population grows too large, the prey begins to die off. This will result in a decrease in the predators. This trend continues as time goes on, implying a stable coexistence of the two populations. The modifed two dimensional Lotka-Volterra predator-prey model also uses a nonlinear system of equations that includes logistic growth of two species, a carrying capacity of the prey, and a predatory factor. In [1] was considered the evolution of a system composed of two deterministic systems of type (1) in random environment. It was shown that in random environment growth rates of the population sizes of the species are bounded above (see [2]). Some results for general case, Lotka-Voltera model with m predators and n preys, was presented in [3].

The main result of the classical Lotka-Volterra model (1) is given in the following theorem:

Theorem 1(see [4]): *Every solution of the Lotka-Volterra system (1) is a closed orbit (except the equilibrium point and the coordinate axes).*

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Since system (1) as solutions has closed orbits, it implies that system (1) has periodic solutions (see [5]).

The modifed two dimensional Lotka-Volterra predator-prey model also uses a nonlinear system of equations that includes logistic growth of two species, a carrying capacity of the prey, a carryng capacity of the predator and a predatory factor. The modifed Lotka-Volterra predator-prey model is given by

$$\begin{cases} \dot{x} = ax - bx^2 - cxy \\ \dot{y} = -dy + exy - fy^2, \end{cases}$$

where a is the growth rate of x, b related to the carrying capacity of x, c is the rate of change of the x due to the presence of y, d is the natural death rate of y, e is the rate of change of y due to the presence of x and f is related to the carrying capacity of y.

The modifed three dimensional Lotka-Volterra predator-prey model also uses a nonlinear system of equations

$$\begin{cases} \dot{x} = ax - bx^2 - cxy - dxz \\ \dot{y} = -ey + fxy \\ \dot{z} = -gz - hzx + ixz + jyz. \end{cases} \tag{2}$$

This system of differential equations modeling the population dynamics of a predator y, a scavenger z, and the prey x, where:

- a is the growth rate of x,
- b is related to the carrying capacity of x,
- c is the rate of change of the x due to the presence of y,
- d is the rate of change of x due to the presence of z,
- e is the natural death rate of y,
- f is the rate of change of y due to the presence of x,
- g is the natural death rate of z,
- h is related to the carrying capacity of z,
- i is the rate of change of z due to the presence of x,
- j is the rate of change of z due to the presence of y.

Using a change of coordinates system (2) can be transformed into system

$$\begin{cases} \dot{x} = x - bx^2 - xy - xz \\ \dot{y} = -by + xy \\ \dot{z} = -cz - dzx + exz + fyz. \end{cases} \tag{3}$$

The Lotka Volterra predator prey model with a scavenger (3) demonstrates the possible population trends when a predator, a prey and a scavenger population interact. It has been shown that the predator and the prey can coexist in the absence of the scavenger, and the scavenger and the prey can coexist in the absence of the predator. However, the scavenger and the predator cannot coexist without the prey. Biologically this is reasonable, because without the prey, the predator will have no food and will die off. The scavenger will then lose all sources of food and will too die off. It has ben shown also that all the three populations can coexist in two ways: they will oscillate between stable populations over time, or the populations will oscillate until they saturate and remain constant over time. System (3) in case where $z=0$, was considered in [6]. One of the main question for system (3) is that if there may be more than two limit cycles (see [7]). These Lotka-Volterra models, as well as some others have been observed in many works.

2. Lotka-Volterra model with two predators

We will consider the Lotka-Volterra model which involves one pray and two predators. The model consists of the following differential equations:

$$\begin{cases} \dot{x} = ax - xy - xz \\ \dot{y} = -by + xy \\ \dot{z} = -cz + xz. \end{cases} \tag{4}$$

where $x(t) \geq 0$ represents prey, $y(t) \geq 0$ and $z(t) \geq 0$ represent predators, and a,b,c are positive parameters. The parameters a, b, c, >0 are interpreted as follows:

-a represents the natural growth rate of the prey in the absence of predators,
-b represents the natural death rate of the predator y in the absence of prey,
-c represents the natural death rate of the predator z in the absence of prey. This problem is presented in [8].

The following theorem characterizes all equilibrium points.

Theorem 2: *System (4) has three equilibrium points:*

- (0,0,0) which is a saddle point,

- (b,a,0) which is a nonhyperbolic point, and

- (c,0,a) which is a nonhyperbolic point,

where $x, y, z \geq 0$ *and a,b,c are positive parameters.*

Proof: Consider the continuous map

$$f(x) = \begin{pmatrix} ax - xy - xz \\ -by + xy \\ -cz + xz \end{pmatrix}.$$

Clearly, $f(x)=0$ at $x_1=(0,0,0)^T$, $x_2=(b,a,0)^T$ and $x_3=(c,0,a)^T$. We first compute the Jacobian matrix of partial derivatives:

$$Df = \begin{pmatrix} a-y-z & -x & -x \\ y & -b+x & 0 \\ z & 0 & -c+x \end{pmatrix}.$$

The derivative at equilibrium point x_1

$$Df(x_1) = \begin{pmatrix} a & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & -c \end{pmatrix}$$

has eigenvalues with different sign, from which we conclude that x_1 is a saddle point (see [9]). So, equilibrium point x_1 is unstable. The derivative at equilibrium point x_2

$$Df(x_2) = \begin{pmatrix} 0 & -b & -b \\ a & 0 & 0 \\ 0 & 0 & -c+b \end{pmatrix}$$

and corresponding eigenvalues are λ_1=b-c, $\lambda_2=i\sqrt{ab}$ and $\lambda_3= -i\sqrt{ab}$. Since eigenvalues λ_2 and λ_3 are pure imaginary, then equilibrium x_2 is a non-hyperbolic point (see [9]). So, we cannot conclude anything at this stage about stability of this equilibrium point. Similarly,

$$Df(x_3) = \begin{pmatrix} 0 & -c & -c \\ 0 & -b+c & 0 \\ a & 0 & 0 \end{pmatrix}$$

and λ_1=c-b, $\lambda_2= i\sqrt{ac}$ and $\lambda_3= -i\sqrt{ac}$, from which we conclude that x_3 is a non-hyperbolic point (see [9]) and we cannot conclude anything about stability of this equilibrium point.

In the next theorem, analyzing the *yz* plane alone, we can predict the behavior of a predator-prey model

$$\begin{cases} \dot{y} = -by \\ \dot{z} = -cz \end{cases} \tag{5}$$

and analyzing *x* axis we can predict the behavior of prey, without predators.

Theorem 3: *If x=0, then for system (5) hold:*

a) $y = y_0\, exp(-bt)\ and\ y \to 0,$ $when\ t \to \infty.$

b) $z = z_0\, exp(-ct)\ and\ z \to 0,$ $when\ t \to \infty.$

c) $If\ y = z = 0, then\ x = x_0 exp(at)\ and$ $x \to \infty, when\ t \to \infty.$

Proof: The proof follows by direct integration.

From Theorem 3 we conclude that both populations will die off in the abscence of the prey. Biologically, this makes sense, because without the prey, the predators will have no food. Thus, the predators populations cannot exist in the absence of the prey.

The next theorem shows that if one predator has bigger than the natural death rate than the other, then it will die off. If predators have equal natural death rate, than they will coexsist.

Theorem 4: Suppose that in the system (4) it holds that y(0)z(0)>0. If

a) b>c, then predator y becomes extinct when t→∞;
b) b<c, then predator z becomes extinct when t→∞;
c) b=c, then predators x and y satisfy the relation y=((y₀)/(z₀))z, for all t.

Proof: If we consider the last two equations of the system (4)

$$\begin{cases} \dot{y} = -by + xy \\ \dot{z} = -cz + xz. \end{cases} \tag{6}$$

and if we divide the first equation of system (6) with *y* and second equation with *z*, then by subtracting them we have

$$\frac{\dot{y}}{y} - \frac{\dot{z}}{z} = c - b.$$

Integrating last equation we have

$$logy - logy_0 - logz + logz_0 = (c-b)t$$

or

$$y = \frac{y_0}{z_0} z\, exp(c-b)t. \tag{7}$$

From (7) we conclude that statements *a*), *b*) and *c*) are true.

Now, we shall consider the behavior of the integral curves of system (4) with respect to the set

$$\omega = \{(x,y,z) \in \Omega | x^2 + y^2 + z^2 \le r^2(t)\}, \tag{8}$$

where r(t) $\in C^1(I, \boldsymbol{R}^+), \boldsymbol{R}^+ = (0,\infty), I \subset \boldsymbol{R}^+$. The boundary surfaces of ω with respect to the set Ω is given by

$$W = \{(x,y,z) \in Cl(\omega) \cap \Omega | \\ H(x,y,z,t) := x^2 + y^2 + z^2 - r^2(t) = 0\},$$

where $Cl(\omega)$ is the set of all points of closure of ω. Let us denote the tangent vector field to an integral curve $(x(t), y(t), z(t), t),\ t \in I$ by

$$T(x, y, z, t) = (ax - xy - xz, xy - by, xz - cz, 1).$$

We will establish some sufficient conditions on the existence and behavior of the classes of solutions of system (4) in a certain region ω, using the retraction method. The vector ∇H is the external normal on surface W. We have

$$\frac{1}{2}\nabla H(x, y, z, t) = (x, y, z, -rr').$$

By means of scalar product $P(x, y, z, t) = \left(\frac{1}{2}\nabla H, T\right)$ on W we shall establish the existence and behavior of integral curves of the system (4) with respect to the set ω. Let us denote with $S^p(I),\ p \in \{0,1,2,3\}$ class solutions $(x(t), y(t), z(t))$ of the system (4) defined on ω which depends of p parameters. We will say that the class of solutions $S^p(I)$ belongs to a set ω if the graphs of functions from $S^p(I)$ are contained in ω. In such a case we write $S^p(I) \subset \omega$. For $p = 0$ we have notation $S^0(I)$ which means that there is at least one solution $(x(t), y(t), z(t))$ on I of system (4) whose graph lies in the set ω. The results of this paper are based on the following lemmas (see [10], [11], [12]), which for the system (4) and set ω, have the form:

Lemma 1: *If it is, for the system (4) the scalar product of* $P(x, y, z, t) < 0$ *on W then the system (4) has a class of solutions* $S^3(I)$ *which belongs to a set* ω, *for every* $t \in I$, *i.e.* $S^3(I) \subset \omega$.
Lemma 2: *If it is, for the system (4) the scalar product of* $P(x, y, z, t) > 0$ *on W then the system (4) has a class of solutions* $S^0(I)$ *which belongs to a set* ω, *for every* $t \in I$ *i.e.* $S^0(I) \subset \omega$.

Let us now consider solution of the system (4).

Theorem 5: *Let function* r(t) $\in C^1(I, \boldsymbol{R}^+)$ *and* $M = max\{a, r\},\ y > 0, z > 0$. If $\big(M - 2(b + c)\big)r < r'$ *on* $Cl(\omega)\cap\Omega$, *then all solutions of system (4) satisfy the condition* $x^2 + y^2 + z^2 \le r^2(t)$ *for* $t > t_0$.

Proof: For the scalar product of $P(x, y, z, t) = \left(\frac{1}{2}\nabla H, T\right)$ on W we have:

$$\begin{aligned} P(x, y, z, t) &= (ax - xy - xz)x + (xy - by)y \\ &\quad + (xz - cz)z - rr' \\ &= (a - y - z)x^2 + (-b + x)y^2 \\ &\quad + (-c + x)z^2 - rr' \\ &\le a^2x^2 + ry^2 + rz^2 \\ &\quad - (b + c)(y^2 + z^2) - rr' \\ &\le \big(M - 2(b + c)\big)r^2 - rr' < 0. \end{aligned}$$

Accordingly, the set of $Cl(\omega)\cap\Omega$ is set of points of strict entrance for the integral curves of the system (4) with respect to set Ω. Now in view of Lemma 1 system (4) has a class of solutions $S^3(I)$ which belongs to a set ω.

Theorem 6: *Let function* r(t) $\in C^1(I, \boldsymbol{R}^+)$ *and* $m = min\{a - 2r, -b - r, -c - r\},\ y > 0, z > 0$. *If* $mr > r'$ *on* $Cl(\omega)\cap\Omega$, *then the system (4) has a one-parameter class of solutions which satisfy the condition*

$$x^2 + y^2 + z^2 \le r^2(t) \ \textit{for}\ t > t_0.$$

Proof: For the scalar product of $P(x, y, z, t) = \left(\frac{1}{2}\nabla H, T\right)$ on W we have:

$$\begin{aligned} P(x, y, z, t) &= (ax - xy - xz)x + (xy - by)y \\ &\quad + (xz - cz)z - rr' \\ &\ge (a - 2r)x^2 + (-b - r)y^2 + (-c \\ &\quad - r)z^2 - rr' \\ &\ge mr^2 - rr' > 0. \end{aligned}$$

Now in view of Lemma 2, system (4) has a one-parameter class of solutions which belongs to a set ω.

3. Conclusion

We have shown that this model has three equilibrium points of which one is unstable, and the stability of the other two points in this part is not describe because they are non-hyperbolic points. In Theorem 3 we have proved that if prey *x* = 0, then predators *y* and *z* are dying and if for predators *y* and *z* hold *y* = *z* = *0*, then the prey *x* tends to infinity.
Also, we are given sufficient conditions for the coexistence of two predators as well as the conditions under which one of the predators became extinct.
Furthermore, we are given sufficient conditions for the existence of solutions for certain classes of Cauchy's solutions of Lotka-Volterra model.
One of the main weakness of this model is unlimited growth of prey *x*, when predators *y* and *z* are equal to zero, which is not biologically realistic.

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