Modified Variational Iteration Method for Sine-Gordon Equation

Umer Saeed

NUST Institute of Civil Engineering,
National University of Sciences and Technology, Islamabad, Pakistan

***Abstract* – In this paper, we introduce a modified variational iteration method for solving nonlinear differential equations. The main advantage of this modification is that it gives stable and relatively accurate results while increasing the domain of unknown function in differential equation, where variational iteration method becomes unstable. The proposed method is based on Chebyshev polynomial approximations in the correction functional of variational iteration method. To show the advantages of the proposed method, we use the sine-Gordon equation as a test problem.**

***Keywords* –Variational iteration method, Chebyshev polynomial, Sine-Gordon equation.**

1. Introduction

It is well known that most phenomena in our world are essentially nonlinear and are described by nonlinear differential equations. In general, it is not possible to find the exact solution of nonlinear differentialequations. Therefore different numerical methods have been applied for providing approximate solutions. Some of these techniques include, the Adomian decomposition method (ADM) [1, 2], the homotopy perturbation method (HPM) [3, 4, 5], the differential transform method (DTM) [6, 7, 8] and variational iteration method (VIM) [9, 10].

DOI: 10.18421/TEM53-09
https://dx.doi.org/10.18421/TEM53-09

Corresponding author: Umer Saeed, NUST Institute of Civil Engineering, National University of Sciences and Technology, Islamabad, Pakistan
Email: umer.math@gmail.com

Variational iteration method [11] was introduced by Jihuan He in 1997 ``as a new approach to nonlinear partial differential equations" in which he implemented variational iteration method on several nonlinear ordinary and partial differential equations. Variational iteration method has been attracted a lot of attention of the researchers for solving nonlinear problems. Variational iteration method does not require a special treatment for nonlinear term as in Adomian decomposition method [12], It directly solves the nonlinear differential equations.

Hemeda [13] uses the variational iteration method for the solution of nonlinear differential equations in which he implemented variational iteration method on two-dimensional nonlinear coupled equations. Wazwaz [14, 15] applies the Adomian decomposition method and variational iteration method for the solution of all type of differential equations, linear or nonlinear, homogeneous or inhomogeneous, with constant or variable coefficients. Also, Biazar [16] implements the variational iteration method for solving nonlinear Burgers' equation. Application of variational iteration method on two-point boundary value problems is discussed by Lu [17] in which he considered the dirichlet boundary conditions.

Consider the sine-Gordon equation [18]

$$u_{tt}-\epsilon^2 u_{xx}+\kappa\sin(u)=0, t>0, x\in[\alpha,\beta], \quad (1)$$
$$u(x,0)=a_1(x), x\in[\alpha,\beta],$$
$$u_t(x,0)=a_2(x), x\in[\alpha,\beta],$$

where ϵ and κ are constants and $a_1, a_2 \in C^\infty[\alpha,\beta]$.Batiha et al. [19] consider the different values of ϵ and κ for finding the numerical solution of sine-Gordon equation by variational iteration method. In [20, 21], authors modify the variational iteration method by using Adomian polynomial for the nonlinear term in sine-Gordon equation. Variational iteration method is also implemented on coupled sine-Gordon equation to get the approximate solution. Matinfar et al. [22] apply the variationaliteration method by using He's polynomials for heat transfer problems in which they introduce He's polynomials in the correction functional. We modified the variational iteration method in such a way that we use the Chebyshev polynomials [23] for approximating the nonlinear term in the correction functional and the proposed method worked well as compared to the variational iteration method. In the present work we apply the Chebyshev polynomial approximations for approximating nonlinear function in the correction functional then we start the iterations for the numerical solution of sine-Gordon equation. The main aim of the proposed method is to get stable and more accurate solution of sine-Gordon equation where the variational iteration method becomes unstable. Illustrative examples show the advantage of our work.

2. Variational Iteration Method and Chebyshev Polynomial Approximation

To clarify the variational iteration method, we consider the following form of differential equation

$$L\,u(t) + N\,u(t) = g(t), \qquad (2)$$

where L, N are linear and nonlinear operators respectively and $g(t)$ is the inhomogeneous term. The variationaliteration method changes the differential equation (2) to a correction functional

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(s)\left(L\,u_n(s) + N\,\widetilde{u_n(s)} - g(s)\right) ds, \qquad (3)$$

where λ is the Lagrange's multiplier which can be obtained via the variational theory, and $\widetilde{u_n}$ is a restricted variation, that is, $\delta\widetilde{u_n} = 0$. According to variational iteration method, first of all find the Lagrange's multiplier optimally, and then select the trial function u_0 to get the successive approximations $u_{n+1}, n \geq 0$,which converge to the exact solution. Consequently, the solution is given by

$$u = \lim_{n\to\infty} u_n. \qquad (4)$$

The Chebyshev polynomial approximation $P_n(u)$ of degree n for f(u) over [-1,1] can be written as

$$f(u) \approx P_n(u) = \sum_{j=0}^{n} C_j T_j(u), \qquad (5)$$

where n is a natural number, $T_j(u)$ are the first kind Chebyshev polynomials which are orthogonal to each other with respect to the weight function

$$w(x) = (1 - x^2)^{-\frac{1}{2}}$$

The Chebyshev nodes are defined as

$$u_k = \cos\left(\frac{\pi(2k+1)}{2n+2}\right), k = 0,1,2,\ldots n. \qquad (6)$$

The coeficients C_j are computed with the formulas [23]

$$C_0 = \frac{1}{n+1}\sum_{k=0}^{n} f(u_k),$$

$$C_j = \frac{2}{n+1}\sum_{k=0}^{n} f(u_k) T_j(u_k),$$

or

$$C_j = \frac{2}{n+1}\sum_{k=0}^{n} f(u_k)\cos\left(\frac{j\pi(2k+1)}{2n+2}\right), j = 1,2,\ldots,n. \qquad (7)$$

Where $T_j(u)$ are the orthogonal Chebyshev polynomials,

$$T_0(u) = 1,$$
$$T_1(u) = u,$$
$$T_2(u) = 2u^2 - 1, \qquad (8)$$
$$T_3(u) = 4u^3 - 3u,$$
$$\vdots$$
$$\vdots$$

In general,

$$T_{n+1}(u) = 2uT_n(u) - T_{n-1}(u), n \geq 1.$$

We take the 3rd degree Chebyshev polynomial approximations $P_3(u)$ for f(u) = sin(u), over [-1; 1], in sine-Gordon equation. As we are considering the 3rd degree Chebychev approximation polynomials so we have $n = 3, k = 0,1,2,3$ and $j = 0,1,2,3$.

$$P_3(u) = C_0T_0(u) + C_1T_1(u) + C_2T_2(u) + C_3T_3(u),$$

where *C's* and *T's* can be calculated from (7) and (8) respectively.

Example: Consider the sine-Gordon equation [21]

$$u_{tt} - u_{xx} + \sin(u) = 0, t > 0, \qquad (9)$$

$$u(x,0) = 0,$$

$$u_t(x,0) = \frac{8}{e^x + e^{-x}}.$$

This is the case when $\kappa = 1$ and $\epsilon = 1$ in (1), the exact solution of (9) is

$$u(x,t) = 4\arctan\left(\frac{2t}{e^x + e^{-x}}\right). \qquad (10)$$

Case 1: Solution by variational iteration method

The correction functional for equation (9) is given as

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^x \lambda(s)\left(u_{ss} - \widetilde{u_{xx}} + \widetilde{\sin(u)}\right)ds. \qquad (11)$$

Making the correction functional stationary, the Lagrange multiplier can be identied as

$$\lambda(s) = (s-t). \qquad (12)$$

This implies that

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^x (s-t)\left(u_{ss} - u_{xx} + \sin(u)\right)ds. \qquad (13)$$

Select the trial function from initial conditions

$$u_0 = \frac{8t}{e^x + e^{-x}}. \qquad (14)$$

MATLAB code for the solution of sine-Gordon equation by variational Iteration Method

```
syms x t s
format long
u = 8 * t/(exp(x) + exp(-x));
for i = 1:1

u = subs(u, t, s);

uss = diff(diff(u, s), s);

uxx = diff(diff(u, x), x);

d = simplify(int(s - t)
        * (uss - uxx + sin(u)), s, 0, t);

u = subs(u, s, t);

u = simplify(((u + d)));

end.
```

We fix the numerical solution at rst iteration

$$u(x,t) \approx u_1(x,t).$$

Case 2: Solution by variational iteration method using Chebyshev polynomial approximations

We apply the Chebyshev approximation for sin(u) in sine-Gordon equation (2) then apply the variational iteration method and investigate the advantage of Chebyshev approximation along variational iteration method.

Consider the 3rd degree Chebyshev polynomial approximations for sin(u), we get

$$\sin(u) \approx A + Bu + Cu^2 + Du^3,$$

where

$A = 1/36028797018963968;$
$B = 35992149818206681/36028797018963968;$
$C = 1/9007199254740992$

and

$D = 1427685159570131/9007199254740992.$

Equation (2) becomes

$$u_{tt} - u_{xx} + A + Bu + Cu^2 + Du^3 = 0, t > 0, \qquad (15)$$
$$u(x,0) = 0, u_t(x,0) = \frac{8}{e^x + e^{-x}}.$$

Applying the variational iteration method on (15), we get

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^x (s-t)\left(u_{ss} - u_{xx} + A + Bu + Cu^2 + Du^3\right)ds. \qquad (16)$$

$$u_0 = \frac{8t}{e^x + e^{-x}}.$$

Consider the first iterate numerical solution

$$u(x,t) \approx u_1(x,t).$$

MATLAB code for solution by variational iteration method using Chebyshev approximations

$syms\ x\ t\ s$
$format\ long$
$u = 8 * \frac{t}{\exp(x) + \exp(-x)};$
$for\ i = 1{:}1$

$u = subs(u, t, s);$

$uss = diff(diff(u, s), s);$

$uxx = diff(diff(u, x), x);$

$$d = simplify\left(int(s-t) * \left(\begin{array}{c} uss - uxx - \\ \frac{1}{36028797018963968} + \\ \frac{35992149818206681}{36028797018963968} * u \\ + \frac{1}{9007199254740992} * u^2 \\ + \frac{1427685159570131}{900719925474099} * u^3 \end{array} \right), s, 0, t \right);$$

$u = subs(u, s, t);$

$u = simplify(u + d);$

$end.$

Figures (1), (2) and (3) indicate the exact solution and numerical solutions by variational iteration method (VIM) and variational iteration method with Chebyshev approximations (VIMC) for problem (9). According to figures (1) and (2) we plot the solutions in three-dimension for $-35 \leq x \leq 35$ and $-50 \leq x \leq 50$ respectively at$0 \leq t \leq 1$, the solution by VIM becomes unstable when x approaches to ± 35 as shown in figure (1)and error increases while x approaches to ± 50, as shown in figure (2) whereas numerical solution by VIMC remains stable for infinite domain. Two-dimensional plotting has also been done to see the behaviour of solutions, we plot the solution in two-dimension by fixing $-37 \leq x \leq 37$ at different values of t, as shown in figure (3). Tables (1) and (2) show the exact solution and numerical solutions by VIM and VIMC along with absolute errors for problem (9) at t = 0.1 and t = 0.5 respectively. Tables (1) and (2) indicate more detail about the solutions which were not clear in the figures. VIMC and VIM represent the solution by variational iteration method along Chebyshev approximations and variational iteration method respectively. Absolute error due to VIM and VIMC are denoted by E2 and E1 respectively. Results in figures and tables conclude that VIM gives more accurate results for small x as compared to VIMC and it becomes unstable for large value of x, whereas VIMC remains stable and accurate for all values of x, as showed in table (1) and (2).

Table 1: Comparison of numerical results by variational iteration method (VIM) and variational iteration method with Chebyshev polynomial approximations (VIMC) with exact solution u(x; t) at t = 0:1

x	t=0.1				
	VIMC	Exact u(x,t)	E1	VIM	E2
1	0.25886	0.25885	9.08232e-7	0.25886	5.37544e-7
3	0.03973	0.03973	7.22484e-8	0.03973	5.14898e-9
5	0.00539	0.00539	9.15012e-9	0.00539	1.305e-11
7	0.72951e-3	0.72950e-3	1.23674e-9	0.72950e-3	1.89598e-14
10	0.36320e-4	0.36320e-4	6.15722e-11	0.36320e-4	2.94210e-13
13	0.18083e-5	0.18083e-5	3.06550e-12	0.18083e-5	2.49735e-11
15	0.24472e-6	0.24472e-6	4.14871e-13	0.24494e-6	2.16912e-10
17	0.33120e-7	0.33120e-7	5.61468e-14	3.18589e-8	1.26061e-9
20	0.16489e-8	0.16489e-8	2.79551e-15	1.25650e-8	1.09161e-8
23	0.82095e-10	0.82095e-10	1.39312e-16	4.27240e-7	4.27158e-7
25	0.11110e-10	0.11110e-10	1.89739e-17	-1.83397e-6	1.83399e-6
27	0.15036e-11	0.15036e-11	2.68783e-18	0e-10	0.15036e-11
30	0.74861e-13	0.74861e-13	2.65688e-19	0.33329e-3	0.33329e-3
33	0.37273e-14	0.37271e-14	1.45096e-19	-1.08543e-3	1.08543e-3
35	0.50455e-15	0.50441e-15	1.39633e-19	0.04921	0.04921

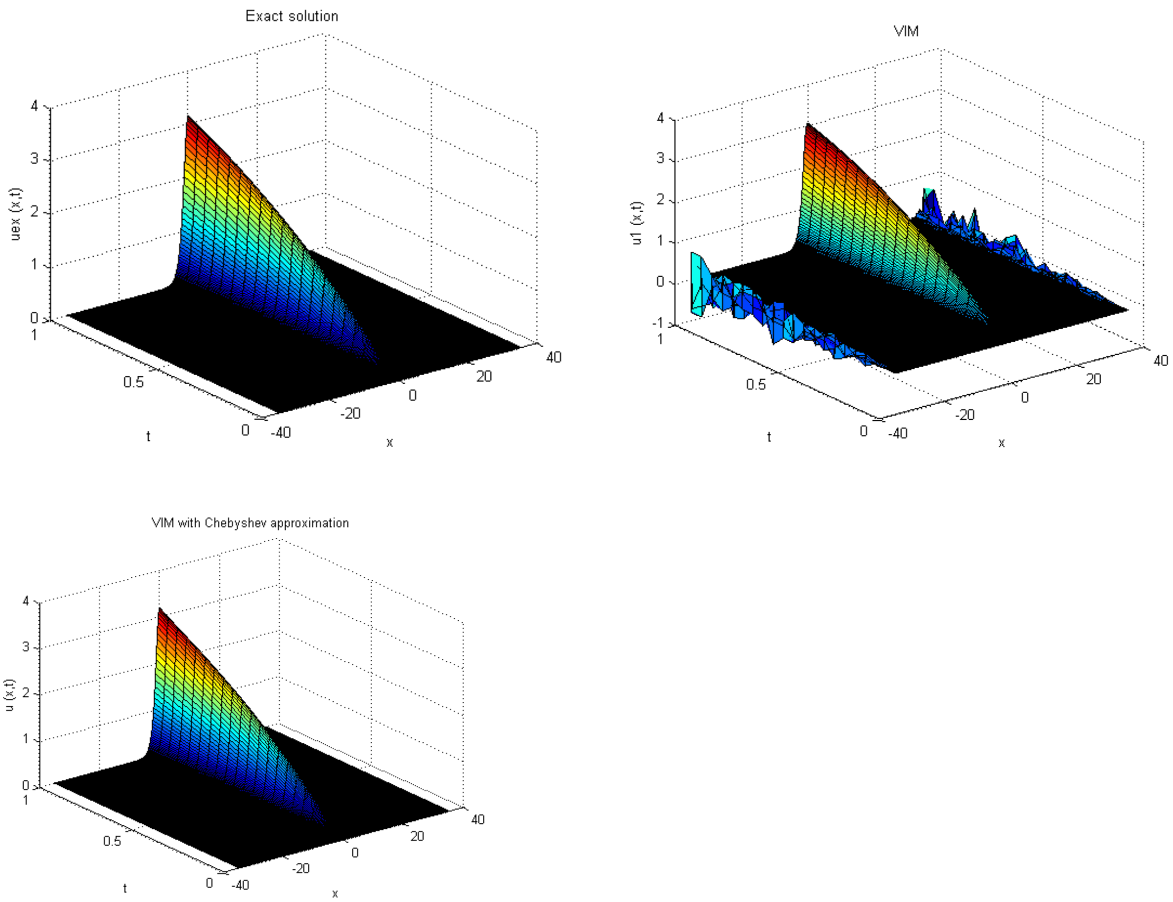


Figure 1: Exact solution, solution by variational iteration method (VIM) and solution by variational iterationmethod with Chebyshev polynomial approximations (VIMC) respectively, $-35 \le x \le 35$*and* $0 \le t \le 1$.

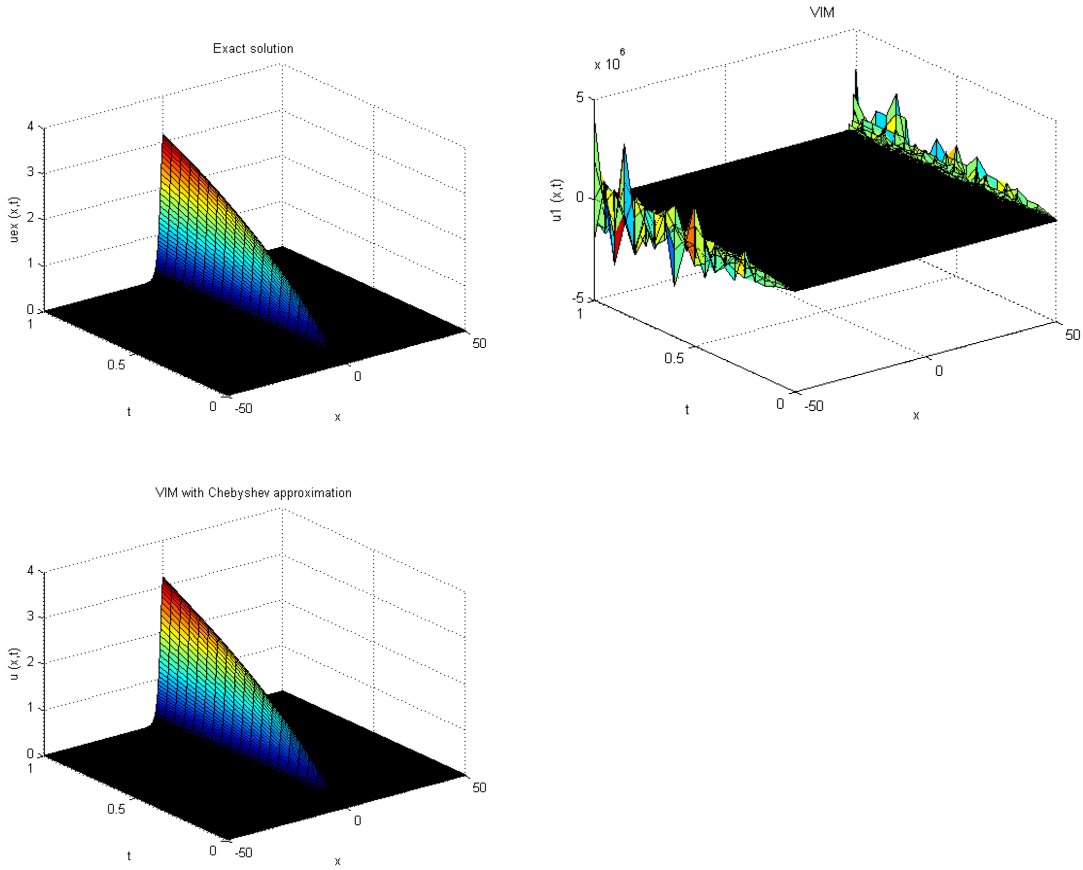
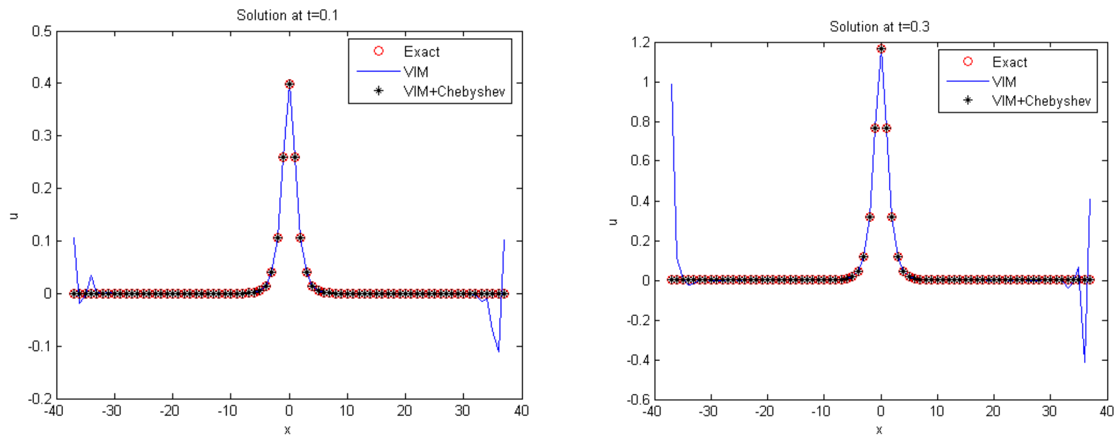


Figure 2: Exact solution, solution by variational iteration method (VIM) and solution by variational iterationmethod with Chebyshev polynomial approximations (VIMC) respectively, $-50 \le x \le 50$*and* $0 \le t \le 1$.



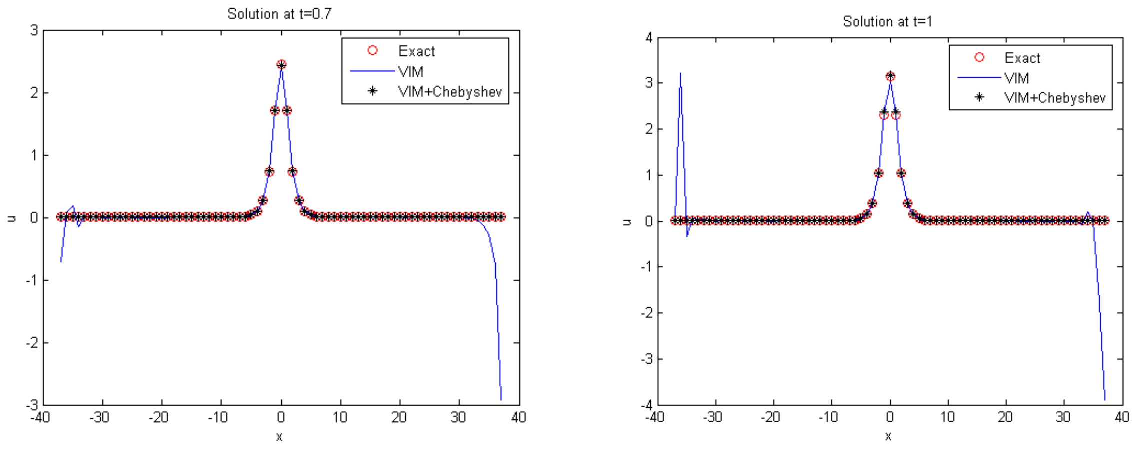


Figure 3: Exact solution, solution by variational iteration method (VIM) and solution by variational iterationmethod with Chebyshev polynomial approximations (VIMC), $-37 \leq x \leq 37$ at different t.

Table 2: Comparison of numerical results by variational iteration method (VIM) and variational iteration method with Chebyshev polynomial approximations (VIMC) with exact solution u(x; t) at t = 0.5.

x	t=0.5				
	VIMC	Exact u(x,t)	E1	VIM	E2
1	1.25512	1.25341	1.70951e-3	1.25511	1.69945e-3
3	0.19852	0.19849	2.37112e-5	0.19851	1.60763e-5
5	0.02695	0.02695	1.180992e-6	0.02695	4.07696e-8
7	0.00365	0.00365	1.54685e-7	0.00365	1.0140e-10
10	1.81607e-4	1.81600e-4	7.69654e-9	1.81600e-4	1.43098e-12
13	9.04170e-6	9.04132e-6	3.83188e-10	9.04151e-6	1.91540e-10
15	1.22366e-6	1.22361e-6	5.18588e-11	1.22395e-6	3.44000e-10
17	1.65605e-7	1.65598e-7	7.01833e-12	1.632647e-7	2.33278e-9
20	8.24496e-9	8.24461e-9	3.49425e-13	-1.61466e-7	1.69710e-7
23	4.10493e-10	4.10475e-10	1.74002e-14	2.10337e-6	2.10296e-6
25	5.55541e-11	5.55518e-11	2.35786e-15	-1.46718e-5	1.46719e-5
27	7.51844e-12	7.51812e-12	3.22101e-16	3.13255e-13	7.20486e-12
30	3.74324e-13	3.74305e-13	1.93332e-17	9.99879e-4	9.99879e-4
33	1.86398e-14	1.86355e-14	4.25926e-18	-3.47338e-2	3.47338e-2
35	2.52562e-15	2.52205e-15	0.21279e-18	0e-14	2.52205e-15

3. Conclusion

VIM is more reliable when we use Chebyshev polynomial approximation in place of trigonometric function of dependent variable in differential equations. The variational iteration method (VIM) and variational iteration method with Chebyshev approximation (VIMC) has been successfully applied for solving sine-Gordon equation. According to comparison of numerical results, mentioned in tables and figures, we conclude that VIM provides stable and accurate results for small x at different t and it goes unstable when x increases, but VIMC still gives stable and accurate results for all x at different t. The proposed method can be easily applied for other differential equation that containtrigonometric functions of dependent variable. The present method can easily be applied to Troesch's Problem, which is the boundary value problem that contains sine hyperbolic function of unknown function.

References

[1] Wu, L., Xie, L.-d., Zhang, J.-M. (2009), Adomian decomposition method for nonlinear differential-difference equations, Communications in Nonlinear Science and Numerical Simulation, 14, 12-14.

[2] A. M. Siddiqui, M. Hameed, Adomian decomposition method applied to study nonlinear equations arising in non-Newtonian flows, Applied Mathematical Sciences, 6 (2012) 4889-4909.

[3] L. Cveticanin, Homotopy perturbation method for pure nonlinear differential equation, Chaos, Solitons and Fractals, 6 (2012) 4787-4800.

[4] A. A. Hemeda, Homotopy perturbation method for solving systems of nonlinear coupled equations, Applied Mathematics and Computation, 208 (2009) 434-439.

[5] Odibat, Z., &Momani, S. (2008), Modified homotopy perturbation method: Application to quadratic Riccati differential equation of fractional order, Chaos, Solitons& Fractals, 36, 167-174.

[6] Do, Y., &Jang, B. (2012), Nonlinear Klein-Gordon and schrodinger equations by the projected differential transform method, Abstract and Applied Analysis, article ID 150527.

[7] Elzaki, T. M. (2012), Solution of nonlinear differential equations using mixture of Elzaki transform and differential transform method, International Mathematical Forum, 7, 631-638.

[8] Taghvafard, H., &Erjaee, G. H. (2010), Two-dimensional differential transform method for solving linear and non-linear Goursat problem, World Academy of Science, Engineering and Technology, 39.

[9] He, J.-H., Wu, G.-C., & Austin, F. (2010), The variational iteration method which should be followed, Nonlinear Science Letters A, 1, 1-30.

[10] Batiha, B. (2012), A numeric-analytic method for approximating quadratic Riccati differential equation, International Journal of Applied Mathematical Research, 1, 8-16.

[11] He, J. (1997), A new approach to nonlinear partial differential equations, Communications in Nonlinear Science and Numerical Simulation, 2 437-440.

[12] Zhu,H., Shu, H., & Ding, M. (2010), Numerical solutions of two-dimensional Burgers' equations by Adomian decomposition method, Computers and Mathematics with Applications, 60 840-848.

[13] Hemeda, A. A. (2012), Variational iteration method for solving nonlinear coupled equations in 2-dimensional space in fluid mechanics, International Journal of Contemporary Mathematical Sciences, 7 1830-1852.

[14] Wazwaz, A. M. (2009), Partial differential equations and solitary waves theory, Higher education press, Beijing and Springer-Verlag Berlin Heidelberg.

[15] Wazwaz, A. M. (2009),The variational iteration method for analytic treatment for linear and nonlinear ordinary differential equation, Applied Mathematics and Computation, 212, 120-134.

[16] J. Biazar, & H. Aminikhah, Exact and numerical solutions for non-linear Burgers' equation by VIM, Mathematical and Computer Modelling 49 (2009) 1394-1400.

[17] Lu, J. (2007), Variational iteration method for solving two-point boundary value problems, Journal of Computational and Applied Mathematics 207, 92-95.

[18] Barone, A., Esposito, F., Magee, C.J., &Scott, A.C. (1971), Theory and applications of sine-Gordon equation, Rev. NuovoCimento 1, 227-267.

[19] Batiha, B., Noorani, M.S.M., &Hashim, I. (2007), Numerical solution of sine-Gordon equation by variational iteration method, Physics Letters A 370, 437-440.

[20] Din, S. T. M., Noor, M. A., &Noor, K. I. (2009), Modifiedvariational iteration method for solving sine-Gordon equations, World Applied Sciences Journal, 6, 999-1004.

[21] Shao, X., &Han,D. (2011), On some modified variational iteration methods for solving the one dimensional sine Gordon equation, International Journal of Computer Mathematics, 88, 969-981.

[22] Matinfar, M., Saeidy, & M., Raeisi, Z. (2011), Modified variational iteration method for heat equation using He's polynomials, Bulletin of Mathematical Analysis and Applications, 3, 238-245.

[23] Mason, J.C., Handscomb, D.C. (2003), Chebyshev Polynomials, Chapman and Hall/CRC.