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Application of Discrete and Fast Fourier Transforms to Increase the Speed of Multiscale Image Analysis

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***Abstract* – The paper compares the accuracy of reconstruction using different wavelets, and the authors also use fast and discrete Fourier transforms together to calculate the forward and inverse continuous wavelet transform in the frequency domain. Due to the use of calculations in the frequency domain, it becomes possible to perform decomposition, reconstruction, image filtering, and other transformations with high performance and precision. For multiscale signal analysis, a wavelet with a rectangular amplitude-frequency response has been constructed, which allows for an increase in the accuracy of decomposition and reconstruction compared to the Mallat algorithm presented in Matlab computer mathematics. At the same time, the time of multiscale analysis is reduced several times compared to the Mallat algorithm.**

***Keywords* – Discrete fourier transform (DFT), fast fourier transform (FFT), fast wavelet transform (FWT), frequency domain, Mallat algorithm.**

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1. Introduction

Currently, multiple-scale analysis (MSA) of signals is performed using discrete orthogonal wavelets - the Mallat algorithm.

Orthogonal wavelets with a compact carrier include the Daubechies (db*N*), Symlet (sym*N*) and Coiflets (coif*N*) wavelets, where *N* is the numeric index (1,2,...), that is, the order of the wavelet. All wavelets are constructed in the time domain. To obtain such wavelets, a system of a large number of equations is solved so that the wavelets have zero moments of a large order. These wavelets enable the analysis of the finer structure of non-stationary signals [1], [2], [3], [4]. These wavelets are not symmetric; to a certain extent, they are close to symmetric wavelets of a Symlet. Due to that asymmetry, the wavelets have nonlinear phase-frequency characteristics which lead to signal distortion during reconstruction. When using such low-order wavelets for decomposition and reconstruction, a mosaic in the image can be noticed if reconstructed without using all the levels.

For the analysis of signals in the frequency domain, it is widely used for image processing [5]; for example, for detecting ships on optical satellite images [6]. There are well-known ways to implement the continuous wavelet transform (WT) using the fast Fourier transform (FFT) [7], [8]. The authors have developed ways to increase the speed of MSA in the frequency domain based on wavelets constructed in the frequency domain [9]. The authors proposed an MSA algorithm using wavelets based on derivatives of the Gauss function [10], [11]. The proposed algorithm enables reconstructing the image. While it is mentioned in the scientific world that the WT is not orthogonal, there is no scaling function, wavelets do not have a compact carrier, and wavelets are used when performing continuous decomposition of signals, the capability of reconstruction is not ensured.

Wavelets based on derivatives of the Gaussian function are symmetric functions, the phase responses of such wavelets are linear . When using such wavelets for decomposition and reconstruction, there is no mosaic in the image, even if it is reconstructed without using all levels.

WT is produced in the frequency domain using FFT. The amount of numerical procedures when using the discrete Fourier transform (DFT) is proportional to the square of the sequence of discrete values of the signal N^2. When using the FFT, the amount of numerical procedures is proportional to the product of a sequence of discrete signal values by the logarithm of this sequence based on two $\log_2 N$, which is significantly less compared to the DFT. This algorithm is called the fast Fourier transform. For example, due to the use of FFT, the calculation time is four orders of magnitude less than with direct numerical calculation of WT for sampling a signal of 32768 samples.

The FFT algorithm, in addition to its advantages, has a number of disadvantages. The first is that the length of the sequence N can only be 2^m, due to the fact that at each m stage the current DFT is defined as a combination of DFTs of half the dimension. The second disadvantage is that it is impossible to calculate the inverse transformation separately for a certain range of Fourier coefficients. For example, it is impossible to calculate only the coefficients from 10 to 20 of a number out of 1024 or to find the sum of only these harmonics. This is because there are two varieties of the FFT algorithm – the time-thinning algorithm, which is called the Cooley-Tukey algorithm [12], [13], [14], and the frequency-thinning algorithm, which is called the Sandy-Tukey algorithm. In order for the output sequence to be linear when using the FFT time-thinning algorithm, the samples of the input sequence should be arranged in binary-inverse order. In order to rearrange the sequence in linear order, in the case of the FFT algorithm by frequency-thinning samples, it is necessary to rearrange the elements with binary-inverse numbers after the conversion, since the frequency sequence is obtained in binary-inverse order, and the input sequence should be arranged in linear order. When using the DFT, both of these disadvantages are absent: there is the possibility to calculate the Fourier coefficients of any *N*-point sequence, and there is the possibility also to calculate the Fourier coefficients of any range separately, without calculating the remaining coefficients, if it is known a priori that they are zero. Such cases often occur, for example, when it is necessary to isolate a useful signal from noise, perform a wavelet analysis of the signal in the frequency domain, and calculate convolution in the frequency domain.

2. Algorithms for Numerical Calculation of Direct and Inverse Fast WT signal in the frequency domain

The algorithm for numerical calculation of the direct fast WT signal $S(t)$ in the frequency domain using wavelets based on derivatives of the Gaussian function is described in [10]. In this algorithm for even wavelets, in this algorithm steps 4 and 7 are not performed. For odd wavelets, steps 3 and 6 are not performed.

Since it is necessary to decompose the image signal into several levels, steps 3-7 must be performed the same number of times. For example, if there are 512 samples in a 512×512 pixel image row, then there will be 9 decomposition levels for one row. For all rows, this number needs to be multiplied by 512, the same for all columns. For a color RGB image, the total number of decomposition levels will be 2754. Unlike Mallat's algorithm, MSA using FFT can be produced with a scale change factor of less than 2, so the number of decomposition levels can be 2, 3, 4 and so on times more. Consequently, it is necessary to find an algorithm that allows you to increase the calculation speed when using FFT. Such an algorithm was created using DFT and presented after the algorithm of numerical calculation of the inverse WT.

The algorithm for numerical calculation of the inverse fast WT signal $S(t)$ in the frequency domain using wavelets based on derivatives of the Gaussian function is described in [10].

3. Application of the Advanced DFT Algorithm for MSA

The authors have developed an algorithm that allows increasing the speed of calculating the DFT by two to three orders of magnitude, depending on the length of the N-point sequence. When performing the Fourier transform, the most time is spent on calculating the sines and cosines of integer multiples of the argument values. In order to reduce the calculation time of sines and cosines, the proposed algorithm uses not a direct calculation, but a recurrence relation, which allows sequentially calculating harmonics using the previous ones. Also, using the periodicity properties of sine and cosine, not all their values are calculated, but only a quarter of *N* and half of all harmonics. That is, by calculating the fourth part of the sine and cosine, all the remaining values can be found. Since *N*/2 sines and *N*/2 cosines (harmonics with different frequencies) are needed to decompose a signal with the number of samples *N* in terms of sines and cosines, then due to the fact that the sines and cosines can be expressed through each other, only half of the harmonics can be used.

Only these values of sines and cosines are used to calculate all the coefficients of the Fourier transformation in a direct and produce the inverse Fourier transform. The speed of the algorithm proposed by the authors in comparison with the FFT algorithm will be less for a single application, since it is necessary to calculate a large value of sines and cosines. However, in the case of repeated use, when the values of sines and cosines are already stored in memory and there is no need to calculate all the Fourier coefficients, the conversion rate is comparable to the FFT rate and higher. Such a case appears when it is necessary to calculate the direct and inverse WT in the frequency domain. The more decomposition levels there are the more Fourier transform operations need to be performed with a different range of Fourier coefficients. This has become important because the authors have designed wavelets with a rectangular frequency response (Figure 1). In frequency response it can be seen that there is no irregularity in the bandwidth in the band of detention and no transition strips. Here one just needs a fast algorithm of DFT, which can be used to signal any *N*-point sequence.

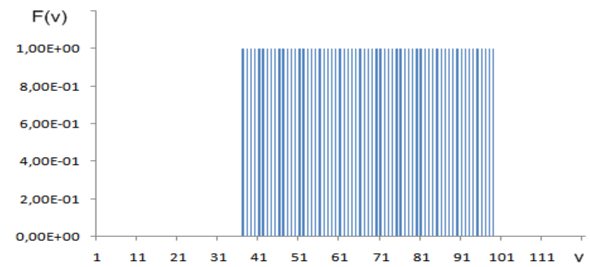


Figure 1. Rectangular frequency response

For greater accuracy, it is desirable that the number of samples $N$is divided into four or one adds 1-3 zeroes to the length of the sequence. If the FFT algorithm is used, it is wasting time to compute the Fourier coefficients, whose value is zero. As levels of decomposition lot, time spent on the calculation of the Fourier coefficients increases. The use of wavelets with a rectangular quartermaster designed in the frequency domain allowed us to reduce the live WT to an image size of 512×512 pixels in 2 times in comparison with the algorithm using wavelets based on derivatives of the Gauss function. The return WT decreased thousands of times, when using a progressive scan image; that is, the computation of the inverse WT time is almost negligible compared to the direct WT. Using an advanced algorithm, DFT has enabled a reduction in the live WT an image size of 512×512 pixels by 50 percent, compared to only using the FFT algorithm for WT.

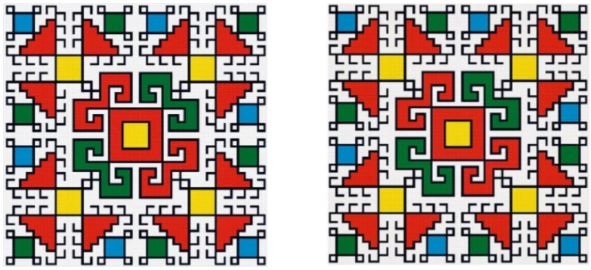
The authors compared the time of decomposition of the images of the developed algorithm in the frequency domain algorithm Mallat introduced in computer mathematics Matlab.

The time decomposition of the image in the algorithm Mallat increases with the order of the wavelets Daubechies (db$N$). Even for the time db2 wavelet decomposition of the image algorithm Mallat, it is more than that of the algorithm in the frequency domain. The computing time of the algorithm Mallat several times increases when it is necessary to cleanse the signal from the noise, because it is necessary to equate some coefficients to zero and then reconstruct the signal. The algorithm Mallat calculations are performed iteratively, and therefore it is necessary to use all levels of the decomposition. And in the MSA algorithm in the frequency domain the time is not growing to clean signal, because it is enough to drop some levels of decomposition and then to reconstruct the signal. The MSA algorithm in the frequency domain computations is not iterative and random. For example, it is possible to decompose the signal into 3, 6, and 7 levels and to restore the signal, and so on. If it is necessary to decompose and reconstruct the signal into these levels many times, then it is possible to decompose it into 3 levels at a time, because it is possible to construct a wavelet that combines 3 wavelets. In fact, it will be a multiband filter. As a result of numerous studies in practice, the authors claim that the method of constructing wavelets can be used to synthesize digital filters with a finite impulse response. When using this method, filters with better characteristics are obtained than when using window methods, frequency sampling, and Chebyshev optimal filters. There are no transition bands on the frequency response for such filters and the shape is strictly rectangular.

4. Comparison of Accuracy of MSA Using Different Wavelets

The use of wavelets with a rectangular frequency response designed in the frequency domain, not only reduces the time the MSA, but also increases the accuracy of the reconstruction of the signal. Figure 2 shows the original and the restored image decomposition and image reconstruction using the wavelet-based derivative of the second order of the Gauss function. Image size 512×512 is decomposed into 36 levels and then reconstructed with the use of all levels. When moving to the next level, the scale factor does not change by 2 times, but by a smaller amount. Because the image is decomposed into 36 levels, we can say that the WT is continuous. A progressive scan is applied horizontally and vertically. When comparing the original and reconstructed images, it can be noticed that they practically do not differ. This is due to the fact that the eye cannot distinguish the intensity values of 255 from 250 in the RGB color model.

To the eye, all the intensities ranging from 250 to 255 for red, green and blue seem to be white. Only at a lower intensity will the color appear darker to the eye. Also, with small intensity values from 0 to 4, everything will appear black, only when the intensity increases to 5-8, the eye sees the image slightly lighter.



a) b)

Figure 2. The original and the reconstructed image using wavelets based on the second derivative of the Gauss function

To demonstrate accuracy, compare the results of the decomposition and reconstruction of single rectangular pulse duration of 36 units for a sample of 1024 samples for different wavelets. Also compare the results of the decomposition and reconstruction 2 rectangular pulses with a difference of amplitudes 10^{15}. Unlike the image, we will be able to see the difference here due to the fact that we are not comparing the light intensity or the change in the waveform on the graph. The slightest change in the signal shape is well noticed by the eyes. Accurate signal recovery in a rectangular shape is the most difficult task, because the spectrum of such a signal is weak and falls off at high frequencies, and for the flat signal, the spectrum is narrower, and therefore such signals are more accurately restored. The rectangular signal is most strongly distorted at the front. The accuracy can be judged by how different the shape of a rectangle is after reconstruction. Next, let us compare the rectangular frequency response wavelets constructed by the authors in the frequency domain with widely used wavelets based on derivatives of the Gauss functions and Daubechies (db$N$).

Figure 3 shows a rectangular pulse after decomposition into 20 levels and subsequent reconstruction with all levels using MHAT-wavelet. In Figure 3 it can be seen that the pulse shape is slightly different from the original rectangular pulse. Where the value of the signal must be exactly equal to zero, a value less than zero to approximately 0.025 mm is, at the front of the pulse – it is less. Also, the value of the momentum is not exactly equal to 1, and it is less than approximately 0.025 mm; that is, the momentum has shifted a bit down. Decomposition of this same impulse to ten levels of deviation from the original pulse becomes even greater. This behavior is caused by the uneven frequency response of the bank of filters (wavelets) being more pronounced when using ten wavelets with different scale factors. For the case of decomposition on 20 levels of scale the factor changes not twice, but on a smaller magnitude in the transition to the next level. From this point of view, this WT is not discrete, and it is continuous. Thus, it is possible to decompose the signal to a much larger number of levels for restoration. The transition to the frequency domain allows us to design wavelets, unlike the Mallat algorithm, which is much more diverse in both Q-factor and resolution, and to use them to study one-dimensional and two-dimensional signals.

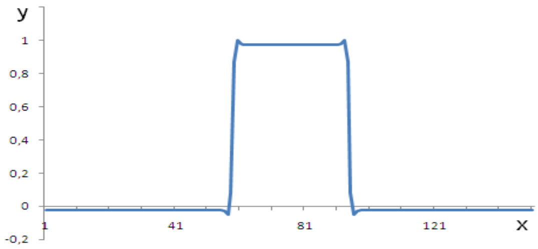


Figure 3. Reconstructed rectangular pulse using the MHAT-wavelet

Figure 4 shows a part of the samples located to the left of the rectangular pulse after decomposition into 10 levels and subsequent reconstruction with all levels using the Daubechies wavelet (db2) in the Mallat algorithm. Since the accuracy in the Mallat algorithm is much higher than when using wavelets based on the MHAT-wavelet, only a part of the left edge of the pulse is presented to estimate the discrepancy with the original. If, as in Figure 3, a pulse with an amplitude of 10^{-15} is also shown, then nothing but a straight line can be seen next to a pulse with an amplitude of 10^{0}. Due to the presence of negative values of the counts, plotting on a logarithmic scale seems to be a difficult task. Figure 4 shows that, as in Figure 3, zeros differ by an average of $2 \cdot 10^{-13}$. With an increase in the order of the Daubechies wavelet, this difference is even greater, for example, for the db40 wavelet; the difference reaches 10^{-6}. Obviously, this is due to the need to solve a large number of equations to obtain zero-order moments for Daubechies wavelets.

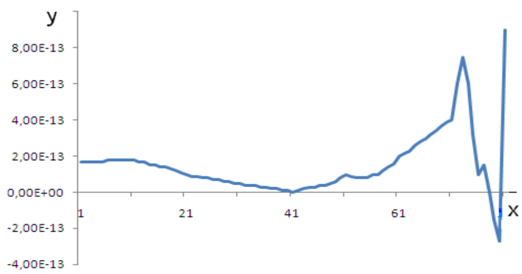


Figure 4. Part of the samples to the left of the rectangular pulse reconstructed using the Daubechies wavelet (db2)

Figure 5 shows part of the samples located to the left of the rectangular pulse after decomposition into 10 levels and subsequent reconstruction with all levels using a wavelet constructed in the frequency domain. Figure 5 shows that the accuracy of the pulse reconstruction is even higher. The pulse exactly repeats the shape of the rectangle if we consider the signal with an accuracy of 10^{-13}. But if we consider the reconstructed signal with even greater magnification, we will see that the zeros also differ by some amount.

Figure 6 shows the same part of the reconstructed signal to the left of the pulse after zooming in. Here, the difference from zero is almost a thousand times less than when processing using the Mallat algorithm, so we can say that the transformation is strictly orthogonal.

We see a pulse whose level is 10^{-15} times less than the neighboring rectangular pulse (Figure 7). This pulse is distorted due to the fact that during decomposition and subsequent reconstruction, the zeros next to the large pulse are not exactly zero, as shown in Figure 6.

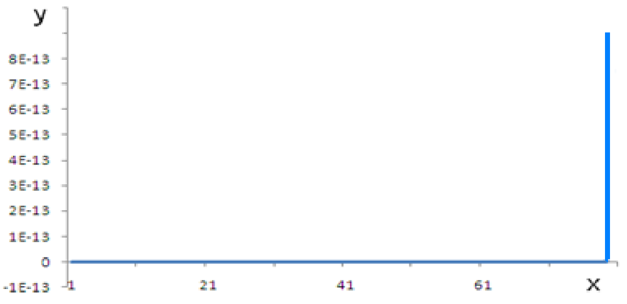


Figure 5. Part of the samples to the left of the rectangular pulse reconstructed using a wavelet constructed in the frequency domain

When reconstructing these two pulses using the Mallat algorithm, a small pulse cannot be seen, since the calculation error is almost 1000 times greater than this pulse. We can say that a small pulse is "drowned" in the noise of the calculation. This is similar to how a sinusoid with small amplitude against a sinusoid with large amplitude can be invisible due to the effect of frequency leakage. When reconstructing two pulses using an MHAT wavelet, a small pulse is visible up to 10^{-13}order. In this case, the reconstructed signal is shifted down, and the shape of the small signal only changes, but does not disappear in noise due to the fact that it is not noise with random values, but an inclined straight line. Against the background of a straight line, a small pulse is noticeable. We can say that the MHAT-wavelet is no worse at restoring a small pulse against a large background than the Daubechies wavelets of a large order.

Figure 6 shows that there are areas near the pulse where the signal value is not exactly zero. In all likelihood, this is also due to the calculation error. It can be seen that the calculation error is almost 1000 times less than when using the Mallat algorithm.

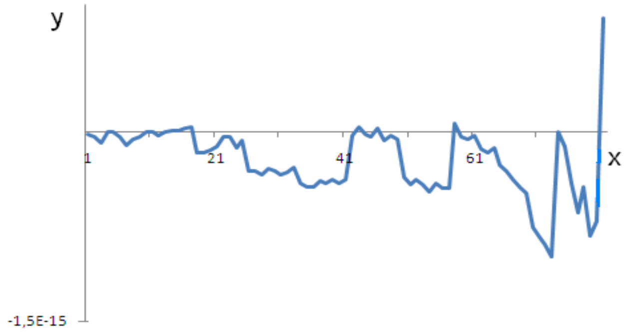


Figure 6. Part of the samples to the left of the rectangular pulse reconstructed using a wavelet constructed in the frequency domain after zooming in

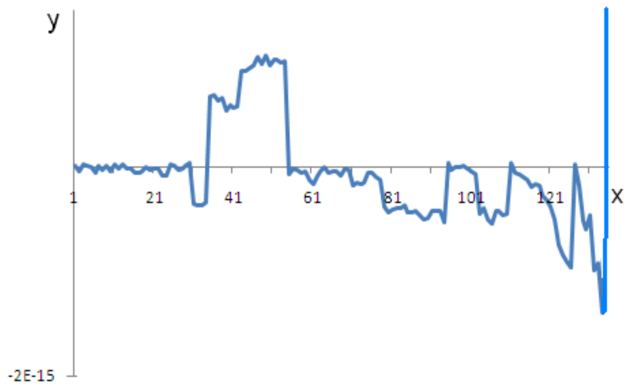


Figure 7. A pulse that is 10^{-15}times smaller than the neighboring rectangular pulse after processing using a wavelet constructed in the frequency domain after zooming in

The use of wavelets with rectangular frequency response for decomposition and reconstruction makes it possible to process one-dimensional and two-dimensional signals more accurately and faster. Such wavelets are much better than the discrete wavelets used in the Mallat algorithm, which is demonstrated in the work.

5. Conclusion

Wavelets based on derivatives of the Gauss function allow reconstructing the signal, conducting multiscale analysis, and filtering one-dimensional and two-dimensional signals, although the scientific literature says the opposite. The Pearson correlation coefficient of the original signal with the reconstructed signal for sampling 1024 samples is at least 0.999. The profiling of the program shows that the time of the wavelet transformation using FFT is 15,000 times less than with direct numerical integration for sampling a signal of 32,768 samples [11], [15].

Strictly orthogonal wavelets constructed in the frequency domain make it possible to reduce the decomposition and reconstruction time of one-dimensional and two-dimensional signals, since they allow for calculation of the Fourier transform in fewer operations. Due to this, the conversion time is reduced even more than when using wavelets based on derivatives of the Gaussian function. They also allow one-dimensional and two-dimensional signals to be reconstructed with greater accuracy. The Pearson correlation coefficient of the original signal with the reconstructed signal for sampling 1024 samples is not less than 0.9999999. The accuracy of the MSA calculation is almost 1000 times higher than for the Mallat algorithm presented in Matlab computer mathematics. This algorithm allows users to get more levels of decomposition because the multiplicity can be less than two. This is the main advantage over the Mallat algorithm for discrete WT. This allows us to study the signal in more detail, since the quality factor of the wavelets is higher. The frequency response of strictly orthogonal symmetric and antisymmetric wavelets has a rectangular shape. To obtain such characteristics in the time domain, it is necessary to solve a system of an infinite number of equations. This method leads to the fact that the calculation time of such equations will increase a lot and will lead to an increase in the calculation error.

Abbreviations

DFT	Discrete Fourier Transform
FFT	Fast Fourier Transform
FWT	Fast Wavelet Transform
MSA	Multiple-Scale Analysis
WT	Wavelet Transform
RGB	Red, Green, Blue
MHAT	Mexican HAT

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