TEM Journal. Volume 11, Issue 4, pages 1924-1929, ISSN 2217-8309, DOI: 10.18421/TEM114-61, November 2022.

Global Stability Analysis of a Certain Second-Order Rational Difference Equation with Nonlinear Terms

Midhat Mehuljić [1], Vahidin Hadžiabdić [1], Jasmin Bekteševć [1], Adnan Mašić [1], Fatih Destović [2]

[1] *Faculty of Mechanical Engineering, Department of Mathematics / University of Sarajevo, Sarajevo, Bosnia and Herzegovina*
[2] *Faculty of Educational Sciences,University of Sarajevo, Sarajevo, Bosnia and Herzegovina*

Abstract **– In this paper we observed the global dynamics and the occurrence of a certain bifurcation for the corresponding values of a certain rational difference equation of the second order with analyzed quadratic terms. The analysis of the local stability of the unique equilibrium point, as well as the unique periodic solution of period two, was performed in detail. The constraint of the equations on both sides for the corresponding values of the parameters is proved and on this basis the global stability is analyzed. The existence of Neimark-Sacker bifurcation with respect to the arrangement of equilibrium points has been proven. Thus, the basins of attraction have been determined in full for all the positive values of the parameters and all the positive initial conditions.**

Keywords **–basins of attraction, difference equation, equilibrium, stability, period two solution.**

DOI: 10.18421/TEM114-61
https://doi.org/10.18421/TEM114-61

Corresponding author: Fatih Destović,
University of Sarajevo-Faculty of Mechanical Engineering, Sarajevo, Bosnia and Herzegovina.
Email: fatih_d@msn.com

Received: 09 September 2022.
Revised: 19 October 2022.
Accepted: 02 November 2022.
Published: 25 November 2022.

1. Introduction

Consider the next difference equation

$$x_{n+1} = \frac{Ax_n^2 + Bx_n x_{n-1} + C}{ax_n^2} \quad (1)$$

where $A, B, C, a > 0$. If we take a shift

$$x_n \to \sqrt[3]{\frac{C}{a}} x_n$$

it can be transformed to

$$x_{n+1} = \frac{\alpha x_n^2 + \beta x_n x_{n-1} + 1}{x_n^2}, \quad (2)$$

where

$$\alpha = \frac{A}{C} \sqrt[3]{\frac{C^2}{a^2}}$$

and

$$\beta = \frac{B}{C} \sqrt[3]{\frac{C^2}{a^2}}.$$

Fractional difference equations of the second order with quadratic terms have been examined in a large number of works, see [1] and [2]. Difference equations of second order with quadratic terms in the denominator or numerator have been studied in many works, see [3] and [4]. For most second-order difference equations with quadratic terms, the mapping associated with the difference equation is monotonic for both variables, see [4], [5], [6], [7], [8]. In these papers, a similar approach to local and global stability analysis was used as we will use for equation (2). This approach to the analysis of global stability will be necessary whenever it is not possible to obtain results in a simpler way, such as the case of using forced linearization and some other approaches given in [9] and [10].

2. Equilibrium Points

Equilibriaof Eq. (2) are the solution of equation

$$\bar{x}^3-(\alpha+\beta)\bar{x}^2-1=0.$$

Let it be

$$g(x)=x^3-(\alpha+\beta)x^2-1.$$

It is easy to check that

$$g(0)=-1 \text{ and } g(+\infty)=+\infty,$$

which implies that function $g(x)$ has a positive root. However,

$$g'(x)=x(3x-2(\alpha+\beta)$$

which implies that $g'(x)$ has two roots

$$x_1=0 \text{ and } x_2=\tfrac{2}{3}(\alpha+\beta).$$

As

$$g(0)g\left(\frac{2}{3}(\alpha+\beta)\right)=\frac{4}{27}(\alpha+\beta)^3+1>0,$$

then Eq.(2) has exactly one positive equilibrium given by

$$\bar{x}=\frac{\lambda}{3}+2\sqrt[3]{2}\frac{\lambda^2}{\mu}+\sqrt[3]{4}\mu$$

where

$$\lambda=\alpha+\beta \text{ and } \mu=\sqrt[3]{2\lambda^3+27+3\sqrt{3}\sqrt{4\lambda^3+27}}.$$

Also , it is useful to note that $g(x)>0$ for $x>\bar{x}$ and $g(x)<0$ for $x<\bar{x}$. As

$$g(1)=-(\alpha+\beta)<0, \qquad g(\alpha)=-\alpha^2\beta-1<0$$

and $g(\beta)=-\alpha\beta^2-1<0$ then we have that

$$\bar{x}>\max\{1,\alpha,\beta\}.$$

3. Analysis of Local Stability

We will investigate local stability of the non-negative equilibrium applying linearized stability Theorem. Set

$$f(u,v)=\frac{\alpha u^2+\beta uv+1}{u^2},$$

then

$$\frac{\partial f}{\partial u}=-\frac{2+\beta uv}{u^3}<0$$

and

$$\frac{\partial f}{\partial v}=\frac{\beta}{u}>0.$$

Let $\bar{x}$ be an equilibrium of (2) and let

$$p=\frac{\partial f}{\partial u}(\bar{x},\bar{x})=-\frac{\beta\bar{x}^2+2}{\bar{x}^3}$$

and

$$q=\frac{\partial g}{\partial u}(\bar{x},\bar{x})=\frac{\beta}{\bar{x}}\in(0,1)$$

denote derivatives of $f(u,v)$ by the variables u and v evaluated in the point $\bar{x}$.

Theorem 1: Let $\bar{x}$ be the equilibrium point of (2).

(i) If $4\alpha^2(\alpha-\beta)>1$ then $\bar{x}$ is sink.

(ii) If $4\alpha^2(\alpha-\beta)<1$ then $\bar{x}$ is unstable saddle point.

(iii) If $4\alpha^2(\alpha-\beta)=1$ then $\bar{x}$ is nonhyperbolic point.

Proof: If

$$\frac{2}{\bar{x}}(\bar{x}-\alpha)=1$$

which is equivalent with

$$\bar{x}=2\alpha$$

then from the facts that

$$(\alpha+\beta)\bar{x}^2+1=\bar{x}^3$$

and $\bar{x}$ is the only positive real root we have that

$$4\alpha^2(\alpha-\beta)=1,$$

hence

$$\frac{2}{\bar{x}}(\bar{x}-\alpha)<1$$

is eqivalent with

$$4\alpha^2(\alpha-\beta)>1.$$

Now it is

$$|p|+q=\frac{\beta\bar{x}^2+2}{\bar{x}^3}+\frac{\beta}{\bar{x}}=\frac{2(\beta\bar{x}^2+1)}{\bar{x}^3}$$
$$=2\left(\frac{(\alpha+\beta)\bar{x}^2+1}{\bar{x}^3}-\frac{\alpha}{\bar{x}}\right)=\frac{2}{\bar{x}}(\bar{x}-\alpha).$$

(i) If $4\alpha^2(\alpha-\beta)>1$

which is equivalent with

$$\frac{2}{\bar{x}}(\bar{x}-\alpha)<1$$

then

$$|p|+q<1,$$

end since

$$q\in(0,1)$$

we have that

$$|p|<1-q<2,$$

which implies that $\bar{x}$ is local asymptotically stable.

(ii) If $4\alpha^2(\alpha-\beta)<1$

which is equivalent with

$$\frac{2}{\bar{x}}(\bar{x}-\alpha) > 1$$

then

$$|p| + q > 1,$$

end since

$$q \in (0,1)$$

we have that

$$|p| > |1-q|,$$

which implies that $\bar{x}$ is unstable saddle point.

(iii) If $4\alpha^2(\alpha-\beta) = 1$

which is equivalent with

$$\frac{2}{\bar{x}}(\bar{x}-\alpha) = 1$$

then

$$|p| + q = 1,$$

end since

$$q \in (0,1)$$

we get

$$|p| = |1-q|,$$

from which we conclude that $\bar{x}$ is nonhyperbolic point.

4. Periodic Solutions of Period-Two and their Stability

Period-two solutions ϕ, ψ, ϕ, ψ satisfy the next system

$$\phi = \frac{\alpha\psi^2 + \beta\phi\psi + 1}{\psi^2} \qquad (3)$$

and

$$\psi = \frac{\alpha\phi^2 + \beta\phi\psi + 1}{\phi^2} \qquad (4).$$

Theorem 2: Eq. (2) has a unique period-two solution which is a sink iff

$$\alpha > \beta \text{ and } 4\alpha^2(\alpha-\beta) < 1.$$

Proof: From (3) and (4) implies that

$$\phi\psi = \frac{1}{\alpha-\beta}$$

and

$$\phi + \psi = \frac{1}{\alpha(\alpha-\beta)}.$$

Hence, for existence of period-two solutions it has to be $\alpha > \beta$. Now, ϕ, ψ are the solutions of the quadratic equation

$$\alpha(\alpha-\beta)x^2 - x + \alpha = 0.$$

Condition, when discriminat of last quadratic equation is positive, is another one condition for existence of period-two solutions, that is

$$4\alpha^2(\alpha-\beta) < 1.$$

Now, one can conclude that if only positive equilibrium is not locally asymptotically stable then there is a prime period two solution, and vice versa. Hence,

$$\phi = \frac{1-\sqrt{1-4\alpha^2(\alpha-\beta)}}{2\alpha(\alpha-\beta)}$$

$$\psi = \frac{1+\sqrt{1-4\alpha^2(\alpha-\beta)}}{2\alpha(\alpha-\beta)}$$

where we assume that $\phi < \psi$. Set

$$u_n = x_{n-1} \text{ and } v_n = x_n, n = 0,1,2, \ldots$$

and write (2) in the equivalent form

$$u_{n+1} = v_n$$

$$v_{n+1} = \frac{\alpha v_n^2 + \beta v_n u_n + 1}{v_n^2}.$$

Let T be a function on $[0,\infty) \times [0,\infty)$ defined by

$$T(u,v) = \left(v, \frac{\alpha v^2 + \beta vu + 1}{v^2}\right).$$

Then (ϕ,ψ) is the fixed point of T^2, second iterate of T. Furthermore, let

$$S = trJ_{T^2}(\phi,\psi)$$

and

$$D = detJ_{T^2}(\phi,\psi),$$

where $J_{T^2}(\phi,\psi)$ is the Jacobian matrix of J_{T^2}, evaluated in (ϕ,ψ). Now, after straightforward calculation we get

$$S = (\alpha-\beta)(2\alpha-\beta)^2 + \frac{\beta}{\alpha}$$

and

$$D = \beta^2(\alpha-\beta).$$

Therefore, by using the facts that $\alpha > \beta$ and $4\alpha^2(\alpha-\beta) < 1$ we have that

$$S - D = 4\alpha(\alpha-\beta)^2 + \frac{\beta}{\alpha} = 4\alpha^2(\alpha-\beta)\frac{\alpha-\beta}{\alpha} + \frac{\beta}{\alpha} < \frac{\alpha-\beta}{\alpha} + \frac{\beta}{\alpha} = 1,$$

and

$$D = \beta^2(\alpha-\beta) < \alpha^2(\alpha-\beta) < 4\alpha^2(\alpha-\beta) < 1.$$

So the two cycle (ϕ,ψ) is locally asymptotically stable because $|S| < 1 + D$ and $D < 1$.

5. Analysis of Global Stability

In this section we have proved the results on global stability of Eq. (2).

By a simple check, we have that it is

$$T(u_n, v_n) = \left(v_n, \frac{\alpha v_n^2+\beta v_n u_n+1}{v_n^2}\right) = (u_{n+1}, v_{n+1})$$

and

$$T^2(u,v) = T\big(T(u,v)\big) = \big(T_{21}(u,v), T_{22}(u,v)\big)$$

where

$$T_{21}(u,v) = \frac{\alpha v^2 + \beta uv + 1}{v^2}$$

and

$$T_{22}(u,v) = \frac{v^4 + \beta v^3(\alpha v^2 + \beta uv + 1) + \alpha(\alpha v^2 + \beta uv + 1)^2}{(\alpha v^2 + \beta uv + 1)^2}.$$

Hence,

$$T^2(u_{2n}, v_{2n}) = T\left(v_{2n+1}, \frac{\alpha v_{2n+1}^2 + \beta u_{2n+1} v_{2n+1} + 1}{v_{2n+1}^2}\right) = (u_{2n+2}, v_{2n+2})$$

which is equivalenth with

$$T^2(x_{2n-1}, x_{2n}) = (x_{2n+1}, x_{2n+2}).$$

The matrix of Jacobian evaluated in the map T has the form

$$J_T(u,v) = \begin{pmatrix} 0 & 1 \\ \frac{\beta}{v} & -\frac{2+\beta uv}{v^3} \end{pmatrix}$$

and

$$\boldsymbol{det} J_T(u,v) = -\frac{\beta}{v}.$$

The matrix of Jacobian of the map T^2 has the form

$$J_{T^2}(u,v) = \begin{pmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{pmatrix},$$

where

$$j_{11} = \frac{\beta}{v},$$
$$j_{12} = -\frac{2+\beta uv}{v^3},$$
$$j_{21} = -\frac{\beta v^4(\beta(\alpha v^2 + \beta uv + 1) + 2v)}{(\alpha v^2 + \beta uv + 1)^3}$$

and

$$j_{22} = \frac{v^2(3\beta + \alpha^2 v^4 + 3\alpha\beta^2 uv^3 + 4v + 5\beta^2 uv + 2\beta uv^2 + 4\alpha\beta v^2 + 2\beta^3 u^2 v^2)}{(\alpha v^2 + \beta uv + 1)^3}.$$

By direct inspection we obtain that the map T^2 is strongly competitive on $[0,\infty) \times [0,\infty)$. Now we have that

$$\boldsymbol{det} J_{T^2}(u,v) = \frac{\beta v^2}{\alpha v^2 + \beta uv + 1}$$

and

$$\boldsymbol{det} J_{T^2}(\bar{x}, \bar{x}) = \frac{\beta \bar{x}^2}{(\alpha+\beta)\bar{x}^2+1} = \frac{\beta}{\bar{x}} > 0.$$

Theorem 3: If $4\alpha^2(\alpha-\beta) > 1$ then Eq. (2) has the positive quilibrium point $E(\bar{x}, \bar{x})$ which is a unique and globally asymptotically stable, where

$$\bar{x} = \frac{\lambda}{3} + 2\sqrt[3]{2}\frac{\lambda^2}{\mu} + \sqrt[3]{4}\mu$$

and

$$\lambda = \alpha + \beta \text{ and } \mu = \sqrt[3]{2\lambda^3 + 27 + 3\sqrt{3}\sqrt{4\lambda^3 + 27}}.$$

Proof: According to Theorem 1 Eq. (2) has equilibrium point which is a unique and also locally symptomatically stable. In addition, for $4\alpha^2(\alpha-\beta) > 1$ Eq. (2) is bounded on the lower and upper side by positive constants.

We have that

$$x_{n+1} = \frac{\alpha x_n^2 + \beta x_n x_{n-1} + 1}{x_n^2} > \alpha$$

and

$$x_{n+1} = \alpha + \frac{\beta x_{n-1}}{x_n} + \frac{1}{x_n^2} < \left(\alpha + \frac{1}{\alpha^2}\right) + \frac{\beta}{\alpha} x_{n-1}.$$

$$x_{2n} < \left(\alpha + \frac{1}{\alpha^2}\right) + \frac{\beta}{\alpha}\left[\left(\alpha + \frac{1}{\alpha^2}\right) + \frac{\beta}{\alpha} x_{2n-4}\right] < \cdots$$
$$< \left(\alpha + \frac{1}{\alpha^2}\right) + \frac{\beta}{\alpha}\left(\alpha + \frac{1}{\alpha^2}\right) + \left(\frac{\beta}{\alpha}\right)^2\left(\alpha + \frac{1}{\alpha^2}\right) + \cdots + \left(\frac{\beta}{\alpha}\right)^n\left(\alpha + \frac{1}{\alpha^2}\right) x_0$$
$$< \frac{\alpha}{\alpha-\beta}\left(\alpha + \frac{1}{\alpha^2}\right) + \left(\frac{\beta}{\alpha}\right)^n\left(\alpha + \frac{1}{\alpha^2}\right) x_0.$$

$$x_{2n-1} < \left(\alpha + \frac{1}{\alpha^2}\right) + \frac{\beta}{\alpha}\left[\left(\alpha + \frac{1}{\alpha^2}\right) + \frac{\beta}{\alpha} x_{2n-5}\right] < \cdots$$
$$< \left(\alpha + \frac{1}{\alpha^2}\right) + \frac{\beta}{\alpha}\left(\alpha + \frac{1}{\alpha^2}\right) + \left(\frac{\beta}{\alpha}\right)^2\left(\alpha + \frac{1}{\alpha^2}\right) + \cdots + \left(\frac{\beta}{\alpha}\right)^n\left(\alpha + \frac{1}{\alpha^2}\right) x_{-1}$$
$$< \frac{\alpha}{\alpha-\beta}\left(\alpha + \frac{1}{\alpha^2}\right) + \left(\frac{\beta}{\alpha}\right)^n\left(\alpha + \frac{1}{\alpha^2}\right) x_{-1}.$$

Note that since it is $4\alpha^2(\alpha-\beta) > 1$ it must be $\alpha > \beta$.

Now that the function f is non-increasing according to the first variable and non-decreasing according to the second variable, and since there are no other equilibrium points or periodic solutions of period two, it follows that each solution with initial conditions from the first quadrant tends to a unique positive equilibrium point. So, the equilibrium point $E(\bar{x}, \bar{x})$ is globally asymptotically stable.

Theorem 4: If $\alpha > \beta > \alpha - \frac{1}{4\alpha^2}$, then Eq. (2) has a unique equilibrium point $E(\bar{x}, \bar{x})$ which is a saddle point and unique solution of period-two $\{P(\phi,\psi), Q(\psi,\phi)\}$ which is globally asymptotically stable, where $\phi = \frac{1-\sqrt{1-4\alpha^2(\alpha-\beta)}}{2\alpha(\alpha-\beta)}$ and

$$\psi = \frac{1+\sqrt{1-4\alpha^2(\alpha-\beta)}}{2\alpha(\alpha-\beta)}.$$

In addition, there is a global stable manifold $W^S(E)$ that is a continuous ascending curve and divides the positive quadrant so that:

(i) Each starting point $(u_0, v_0) \in W^S(E)$ tends to the point E.
(ii) If $(u_0, v_0) \in W^+(E)$ (the area under the stable manifold) then the subset of even members (u_{2n}, v_{2n}) tends to the point Q, and the subset of the odd members (u_{2n+1}, v_{2n+1}) is attracted to the point P.
(iii) If $(u_0, v_0) \in W^-(E)$ (the area above the stable manifold) then the subset of even members (u_{2n}, v_{2n}) tends to the point P, and the subset of the odd members $(u_{2n+1}, v_{2n+1})$is attracted to the point Q.

Proof : It is not difficult to see that Eq. (2) for given parameter values constrained from the bottom and top by positive constants. According to Theorem 1 for $\alpha > \beta > \alpha - \frac{1}{4\alpha^2}$ Eq. (2) has a unique positive equilibrium point which is a saddle point. Furthermore the mapping T in the area $R \cap int(Q_1(E) \cup Q_3(E))$ has no fixed points or periodic points of period two. Simply check, it holds that $\det(J_T(E)) < 0$ and $T(x) = \bar{x}$ only when $x = \bar{x}$. Now, since the map T is competitive and T^2 is strictly competitive, the proof of the ours theorem follows on the basis of Theorem 10 in [11].

Theorem 5: If $\beta > \alpha$, then Eq. (2) has a unique positive equilibrium point $E(\bar{x}, \bar{x})$ which is a saddle point and there are no periodic solutions of period two. Furthermore, there is a stable manifold $W^S(E)$ which is a continuous ascending curve dividing the first quadrant and is valid:

(i) Each starting point $(u_0, v_0) \in W^S(E)$ is attracted to the saddle point E.
(ii) If $(u_0, v_0) \in W^+(E)$ (the area under the stable manifold) then the subset of even members (u_{2n}, v_{2n}) tends to the point (∞,α) and the subset of the odd members (u_{2n+1}, v_{2n+1}) is attracted to the point (α, ∞).
(iii) If $(u_0, v_0) \in W^-(E)$ (the area above the stable manifold) then the subset of even members (α, ∞) tends to the point P, and the subset of the odd members (u_{2n+1}, v_{2n+1}) tends to the point (∞,α).

Proof: The proof of this theorem proceeds quite similarly to the proof of Theorem 4, taking into account the fact that for $\beta > \alpha$ Eq. (2) is bounded only from the bottom by α and there are no periodic solutions of period two.

Theorem 6: If $\beta = \alpha - \frac{1}{4\alpha^2}$ then Eq. (2) has a unique nonhyperbolic equilibrium point $E(2\alpha, 2\alpha)$. Also, there is a continuous ascending curve C which is a subset of basins attraction of the equilibrium point $E(2\alpha, 2\alpha)$ and divides the first quadrant so it's worth it:

i) Each starting point from curve C tends to the equilibrium point $E(2\alpha, 2\alpha)$.
ii) ii) For each starting point $(u_0, v_0) \in R \setminus C$, sequence (u_n, v_n) tends to the equilibrium point $E(2\alpha, 2\alpha)$.

Proof: Eq. (2) has a unique positive nonhyperbolic equilibrium point $E(2\alpha, 2\alpha)$. All conditions for the existence of a continuous ascending curve C are satisfied, see [12]. For every point $(u, v) \in W^-(E)$ that is above the curve C $\exists n_0 \in N$ such that $T^n(u, v) \in intQ_2(\bar{x})$ for $n \geq n_0$. Similarly, for every point $(u, v) \in W^+(E)$ that is below the curve C $\exists n_0 \in N$ such that $T^n(u, v) \in intQ_4(\bar{x})$ for $n \geq n_0$, see[11].
Let it be

$$S(t) = \frac{\left(\frac{1}{4\alpha^2} - \alpha\right)t + \sqrt{\left(\frac{1}{4\alpha^2} - \alpha\right)^2 t^2 - 4(\alpha - t)}}{2(\alpha - t)}.$$

The following arrangement is now true
$(t, S(t)) \leq_{SE} E$, for $t < \bar{x}$, and $E \leq_{SE} (t, S(t))$ for $t > \bar{x}$.

Also we have that holds

$$T^2((t, S(t)) = \frac{1}{32t^2(t-\alpha)\alpha^4}\Big(32t\alpha^4 - 32\alpha^5 + 32t^3\alpha^5 + t\alpha^2\sqrt{d} - 4t\alpha^5\sqrt{d} + t^2\big(1 - 8(\alpha^3 + 2\alpha^6)\big)\Big),$$

where

$$d = \frac{64t\alpha^4 - 64\alpha^5 + (t - 4t\alpha^3)^2}{\alpha^4}.$$

Now we have that

$$T^2\big(t, S(t)\big) \leq_{SE} \big(t, S(t)\big) \text{ for } t > \bar{x}$$

and

$$\big(t, S(t)\big) \leq_{SE} T^2\big(t, S(t)\big) \text{ for } t < \bar{x}.$$

This is true because

$$\frac{1}{32t^2(t-\alpha)\alpha^4}\Big(32t\alpha^4 - 32\alpha^5 + 32t^3\alpha^5 + t\alpha^2\sqrt{d} - 4t\alpha^5\sqrt{d} + t^2\big(1 - 8(\alpha^3 + 2\alpha^6)\big)\Big) - \frac{\left(\frac{1}{4\alpha^2} - \alpha\right)t + \sqrt{\left(\frac{1}{4\alpha^2} - \alpha\right)^2 t^2 - 4(\alpha - t)}}{2(\alpha - t)} > 0,$$

which is true if and only if $t > \bar{x}$.

By the properties of monotony $t < \bar{x}$ implies that $T^{2n}\big(t, S(t)\big) \to E$ when $n \to \infty$. Similarly, $t > \bar{x}$ implies that $T^{2n}\big(t, S(t)\big) \to E$, when $n \to \infty$.

If $(\tilde{u},\tilde{v}) \in intQ_2(\bar{x})$, $\exists\ \tilde{t}$ such that

$$(\tilde{t},S(\tilde{t})) \leq_{SE} (\tilde{u},\tilde{v}) \leq_{SE} E.$$

By the properties of monotony of map T^2 we have that

$$T^{2n}(\tilde{t},S(\tilde{t})) \leq_{SE} T^{2n}(\tilde{u},\tilde{v}) \leq_{SE} E$$

which implies that

$$T^{2n}(\tilde{u},\tilde{v}) \to E$$

and

$$T^{2n+1}(\tilde{u},\tilde{v}) \to T(E) = E,$$

when $n \to \infty$.

If $(\bar{u},\bar{v}) \in intQ_4(\bar{x})$, $\exists\ \bar{t}$ such that

$$E \leq_{SE} (\bar{u},\bar{v}) \leq_{SE} (\bar{t},S(\bar{t})).$$

By the properties of monotony of map T^2 we have that

$$E \leq_{SE} T^{2n}(\bar{u},\bar{v}) \leq_{SE} T^{2n}(\bar{t}\ ,S(\bar{t}))$$

which implies that

$$T^{2n}(\bar{u},\bar{v}) \to E$$

and

$$T^{2n+1}(\bar{u},\bar{v}) \to T(E) = E,$$

when $n \to \infty$.

This proves that in the case when $\beta = \alpha - \frac{1}{4\alpha^2}$ each solution tends to unique hyperbolic equilibrium point $E(2\alpha, 2\alpha)$.

6. Conclusion

For difference equation (2), which was observed for positive parameter values and positive initial conditions, local and global asymptotic stability were fully examined and attraction basins were found. The bifurcation of doubling period was determined, and the absence bifurcation of Neymark-Sacker. In case of limited solution Eq. (2) from the lower and upper side of the positive constants we see that each solution tends to either a unique equilibrium point or a periodic solution of period two when it exists. Great benefit in a complete description of the dynamics of Eq. (2) was obtained due to the monotonicity of the map T^2. Note that the differential equation

$$x_{n+1} = \alpha + \frac{1}{x_n^2}$$

which is obtained from Eq. (2) when $\beta = 0$, although the first order has interesting dynamics. The same is true for the differential equation of second order

$$x_{n+1} = \frac{\beta x_n x_{n-1} + 1}{x_n^2},$$

which is obtained from Eq. (2) when $\alpha = 0$.

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