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Axial Symmetry in Non-standard Problems with Connection to Musical Art

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***Abstract* – In the paper, we will focus on the relationship between the teaching subjects of mathematics and musical art from the point of view of solving non-standard geometric problems. We designed several problems in order to point out the connection between theoretical knowledge and practical life. We present interdisciplinary relations as one of the possible innovative forms of teaching in secondary school.**

***Keywords* – interdisciplinary relationships, axial symmetry, musical art, mathematical problems, evaluation of research.**

1. Introduction

An inseparable part of mathematical education is geometry. Despite the fact that there are many people who do not admit, geometry is very essential for our everyday life. It belongs to an important factor for the development of imagination, creativity, but also for various professions. Geometry should not be neglected in the mathematics curriculum.

Another significant factor for a successful educational process is the connection of knowledge in different teaching subjects.

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The independence of teaching subjects currently causes a certain isolation between them. A complex understanding of phenomena and processes within the teaching subjects leads pupils to think in certain contexts.

Through the positive influence of interdisciplinary relationships pupils learn the subject matter in the necessary depth, and therefore they can use their knowledge even after a long time.

In the paper we focused on the connection between mathematics and music in problems that are not standard for the educational process.

We define interdisciplinary relations as relations between concepts, phenomena, events, situations and their transfer into teaching subjects. Their application enables pupils to understand the natural and social reality as a whole and uniformly [1].

The teacher can influence the application of interdisciplinary relationships only in the subject he teaches. In the class, he can determine only a strategy to bring interdisciplinary relationships closer. In [2] authors divide strategies into three groups:

- *putting into context* – clarifying the subject in the context of history, culture and personal experience, pupils acquire theoretical, methodological, epistemological and historical relationships between subjects,
- *creation of a concept* – identification of the main area common to two or more subjects, understanding of important natural rules that apply without human intervention, pupils transform empirical information into general knowledge, the role of the teacher is very important in this strategy,
- *focusing on the problem* – pragmatic and real life is verified, concepts, procedures and ideas from different subjects must be used when solving problems, the aim is to create a real solution or product, the aim is to use the means of different teaching means of different teaching subjects in solving a challenging problem.

Problems with different demands on cognitive functions are applied in the teaching of mathematics in each of its phases, at each level and type of school.

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Through mathematical problems, pupils learn mathematics and problem solving methods. Mathematical problems fulfil various aims, which are ensured through their solutions, as follows:

- *educational goals* – problems form mathematical knowledge, skills, habits and views on mathematical topics that pupils can use,
- *educational goals* – problems lead to independence and creativity, shape general abilities,
- *developing goals* – problems develop thinking oriented towards effective mental activity, critical thinking, support personal and social development [3].

The group of non-standard problems includes problem, project and application word problems, but learning word problems belong to standard mathematical problems. In [4] is stated that suitable non-standard problems for the educational process must meet the following requirements:

- They provide pupils with a real choice, they are a challenge for them to solve.
- They are connected to, based on and develop the core curriculum.
- They provide pupils with resources from real situations, they show the connection of key subjects with life.
- The time dedicated to them should be used effectively, the pupils' work should be aimed at asking questions and finding the right answer.
- They are oriented towards direct experience and provide pupils with a structure in which they can find support in the search for knowledge, skills and role models in real life.
- For teachers, they are the building blocks of a teaching unit and for choosing teaching methods.

In the following part, we present the problems proposed by us and the statistically processed results of solving the given problems by secondary school pupils.

2. Methodology of the Research

Specifically, we assigned pupils to solve problems about congruent transformations in the plane, in which we used only axial symmetry. Geometric problems were linked to a piano score, an organ score or violin parts. We know that pupils often use only basic geometric figures in mathematics lessons. Pupils don't think they can use unconventional shapes in geometry.

We assigned the following problem A, problem B and problem C to the pupils (the age of these pupils was 15 – 17 years). Then we selected 38 solutions for a more detailed analysis.

Problem A

In Figure 1. you can see one bar of notes, but the notes in the second notation (for the left hand) are missing. Therefore, to the notation draw the notes that you get by axial symmetry with the axis o_1, o_2, which will be the missing notes for the left hand.

Figure 1. Problem A

Problem B

In this problem, an excerpt from the song "We call to the Madonna" is used, in which we marked two axes of symmetry o_1, o_2. In Figure 2, mark the notes that are symmetrical along the given axes. Find the subject and also the image of symmetrical notes. Alternatively, if you see another congruent plane transformation in the notation, mark it and name it.

Figure 2. Problem B

Problem C

Figure 3. shows half of the new violin bridge without the string notches. Draw the second part of this violin bridge if you know that it is symmetrical along the axis *o*.

Figure 3. Problem C

Didactic Variables of Pupils´ Strategies in Problems

We analyzed pupils' steps in strategies in every problem by statistical implicative analysis methods. We used C.H.I.C. statistical software (Classification Hiérarchique Implicative et Cohésitive) for better analysis of relationships between stated didactical variables.

In following lines, we present didactic variables, which were listed in the a-priori analysis.

- Type A – pupils' strategies of problem A:
  A1 – pupils solved the step 1,
  A2 – pupils solved the step 2,
  A3 – pupils solved at least one-part step,
  A4 – pupils solved problem A.
- Type B – pupils' strategies of problem B:
  B1 – pupils solved the step 1,
  B2 – pupils solved the step 2,
  B3 – pupils solved the step 3,
  B4 – pupils solved the step 4,
  B5 – pupils solved at least one-part step,
  B6 – pupils solved problem B.
- Type C – pupils' strategies of problem C:
  C1 – pupils solved the step 1,
  C2 – pupils solved the step 2,
  C3 – pupils solved the step 3,
  C4 – pupils solved the step 4,
  C5 – pupils solved problem C.

In the first problem A, it was necessary to display the notation head using axial symmetry. However, the pupils had to be careful, and it was necessary to notice that the axis o_1 was drawn for the first half bar and another axis o_2 was drawn for the second half bar. Correctly step A1, we considered the correctly completed notation head in the first half bar. Notation heads were symmetrical about the axis o_1. As correctly step A2, we marked notation head in the second half bar same as to step 1 but axis o_1 was replaced by axis o_2. The situation in which the pupil to solved at least one-part step in problem A was marked as A3. When the pupil to solve the complete problem A, we marked the situation as A4. Figure 4. shows steps A1 and A2, which belong to the said bars.

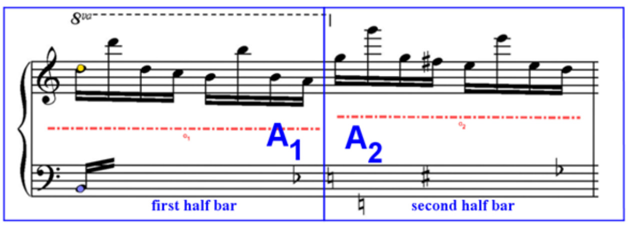


Figure 4. Excerpt from the Czerny's etude [5]

Problem B was interesting because it involved two types of steps. The first type of step was aimed at designating such a notation head that was axially symmetric. This assignment occurred in the first and third bars. We identified the solution to this step of the problem by B1 and B2. The second type of step was, that the correct solution for the second and fourth bars is that it should be nothing marked. We called this step of the variables B2 and B4. We also used the designation B5, which means a situation in which the pupil correctly solved at least one-part step in problem B. We also evaluated the situation when the pupil solved the complete problem B. We can also see the assignment with the designation of individual steps in Figure 5.

Figure 5. Excerpt from We call to the Madonna [6]

The third problem C was focused on displaying the axial symmetry of the violin bridge, which means non-standard shapes. Representation in the axial symmetry displaying segments and a circular arc, which we can see in green in Figure 3., we marked by C1. We have also used the C2, C3, and C4 markings for those pieces of violin bridge that are drawn also in the Figure 6.

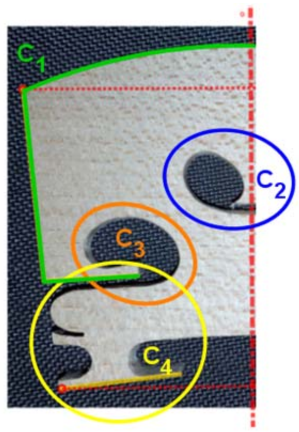


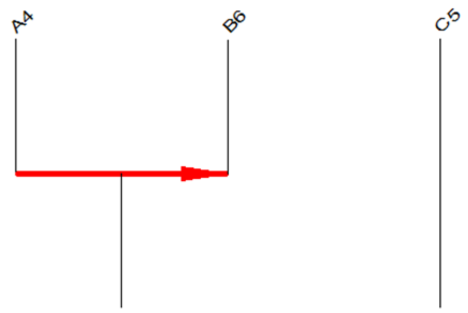
Figure 6. Half of a violin bridge

3. Evaluation of the Research

The statistical program C.H.I.C was used to analyze the association of variables related to axial symmetry in non-standard tasks with a link to musical art. There were used mentioned didactic variables based on which they were compiled implicative trees.

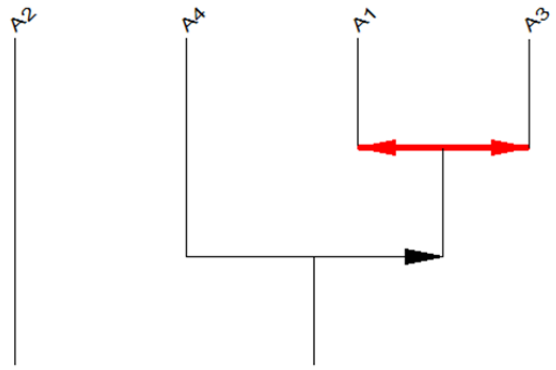
There is the implicative tree for complete right solutions to three non-standard problems. Graph 1. shows the significant statistical implications between a complete correct solution to problem A and a complete correct solution to problem B (implication A4 and B6). There is a significant connection between the variables A4 and B6 (cohesion = 0.947).

It means that a pupil who was able to solve problem A correctly solved problem B, too. This graph also shows that there is no connection between the correct solution of problem C at the same time problem A or problem B.



Graph 1. Implicative tree for problem A, problem B, problem C

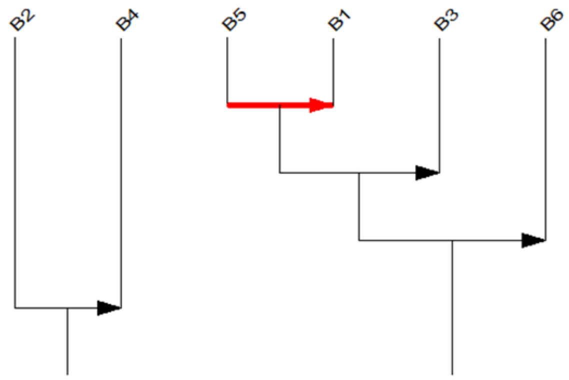
Interesting results are shown in Graph 2. As we can see from the graph, there is a significant statistical two-way implication between them. The implication is between the correct insertion of the header in the first half of the bar and at least one correct header (implication A1 and A3 and at the same time implication A3 and A1). This implication (cohesion = 1) shows that if the pupil correctly identified at least one notation head, it was just and only the note from the first half of the bar. Graph 2. also shows that there is no connection between A2 and other variables of problem A.



Graph 2. Implicative tree from the variables of problem A

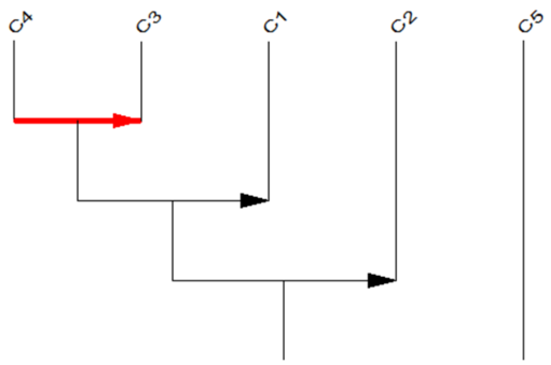
Graph 3. points the significant statistical implications between two parts of problem in bars. In the first part of problem pupils had to find and mark axially symmetrical notation heads. The second part of problem was made by bars, which did not contain axially symmetrical notation heads. Our expected answer is that pupils do not fill in the tact notes in the notation.

The strongest hierarchy is between the variables B5 and B1 (cohesion = 1). This means that if the student has correctly solved at least one part of the step (B5), they have solved the first bar in the problem, which is listed as variable B1. Then the sequence of implications (((B5 → B1) → B3) → B6) is lower but still significant. It means, if pupil correctly marked notations heads in bar B1, so he correctly marked notations heads in bar B3 (cohesion = 0.997). Although there is an implication between variables B2 and B4 (cohesion = 0.960), it is interesting that between implication (B2 → B4) there is no connection of variables with the other mentioned variables for problem B. It only means that if the pupil did not mark any notation head correctly in the second bar, then he did not mark any notation head correctly in the fourth bar.



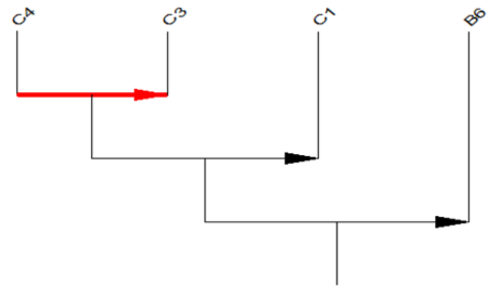
Graph 3. Implicative trees from the variables of problem B

In Graph 4. We can see connection between variables C4 and C3 (cohesion = 1). There is also a strong implication (C4 → C3) → C1, (cohesion = 0.998). Somewhat smaller but still significant is the implication ((C4 → C3) → C1) → C2, (cohesion = 0.944). This problem is special that there does not exist a connection between the implications of displaying pieces of violin bridge and a correct displaying of the whole violin bridge. In Figure 3., we see that variables C3 and C4 are connected to each other. We think that if the pupil didn't solve C4 correctly, he didn't solve C3 either because these variables follow immediately after each other.



Graph 4. Implicative tree for problem C

There is also a Graph 5. that shows the interaction between variables of the different problems. From this graph can be seen that if the pupil solved problem C in the interaction (C4 → C3) → C1 (cohesion = 0.998) he also solved the whole problem B (C4 → C3) → C1) → B6 (cohesion = 0.983). There we see that the pupil who correctly solved the most difficult parts of problem C, that is variables C3 and C4, also completely solved problem B.



Graph 5. Implicative tree for steps of problem C and complete problem B

We see the implicative graph displayed in Graph 6 and the significant implication is confirmed on this graph. It is implication similarly as we bar see on Graph 2., Graph 3. and Graph 4, too. The implicative graph confirmed what implicative tree showed.

Red arrow displays the best significant implication (cohesion ≥ 0.99) from variable A1 to variable A3, from variable B5 to variable B1, also from variable C4 to variable C3 and from variable A4 to variable B2. Blue arrows illustrate cohesion more than 0.96 and green arrow represent cohesion more than 0.94. We can see also in the Graph 6. that the red arrow creates interesting sequences of variables.

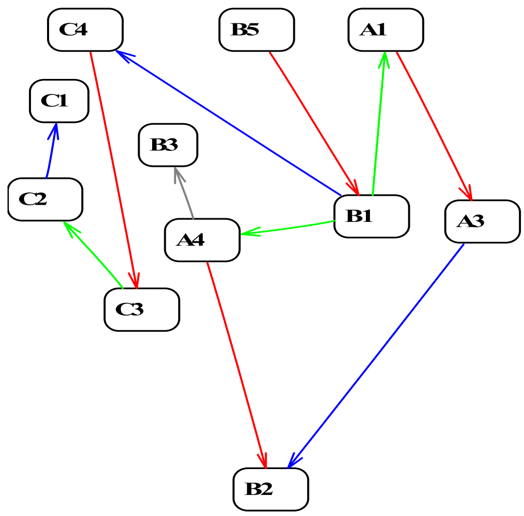
The variables in the implications on the Graph 6. create as if two groups. One of them is created only by variables of problem C. In the second group we can see implications between variables of problem A and problem B.

Variables of problem C follow in a separate sequence with different percentages. If the pupils solved the most demanding part correctly (variable C4), they solved the problem specified in the variable C3 as well (C4 → C3: cohesion ≥ 0.99). We also see that if the pupils solved problem specified in the variable C3 correctly, they also solved problem specified in the variable C2 (C3 → C2: cohesion ≥ 0.94 and then also correctly solved problem specified in the variable C1 (C2 → C1: cohesion ≥ 0.96).

The Graph 6. shows that the interesting point is variable B1 which means the correct solution of the first bar in problem B. We can see that three important strategies based on variable B1 were used. One of them showed that if the pupils correctly solved the first bar of problem B, they also correctly solved the first half of the bar of problem A (B1 → A1: cohesion ≥ 0.94). The strategy proceeds so that if the pupil correctly solved the first half of the bar, it is certain that he correctly identified at least one notation head of problem A (A1 → A3: cohesion ≥ 0.99). If the mentioned group of pupils used this strategy, they were also successful in solving the second bar of the problem B (A3 → B2: cohesion ≥ 0.96).

The correct solution of the second bar of problem B seems to be the endpoint of another implication. The implication is based on the already mentioned variable B1, it does not follow the described way. If pupils correctly solves the first bar of problem B he also will solve the whole problem A (B1 → A4: cohesion ≥ 0.94) and then solved the second bar of the problem B as well (A4 → B2: cohesion ≥ 0.99). It means, if pupil correctly marked notation heads in first bar, so they correctly solved second bar, that is, they did not mark anything.

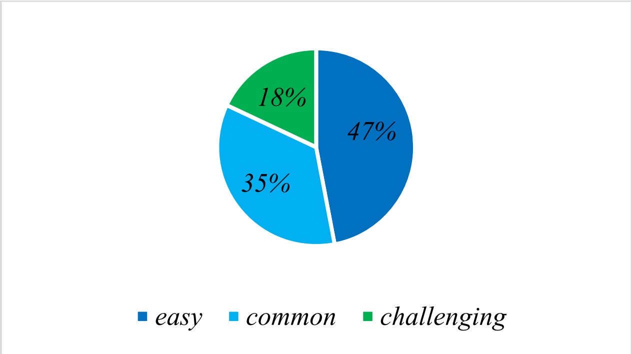
The mentioned two groups of implications are linked by implication between variable B1 and variable C4. It means that if pupils correctly solved the first bar of problem B so they correctly solved the most demanding part of the problem C as variable C4. Cohesion of implication between variable B1 to variable C4 is more than 0.96.



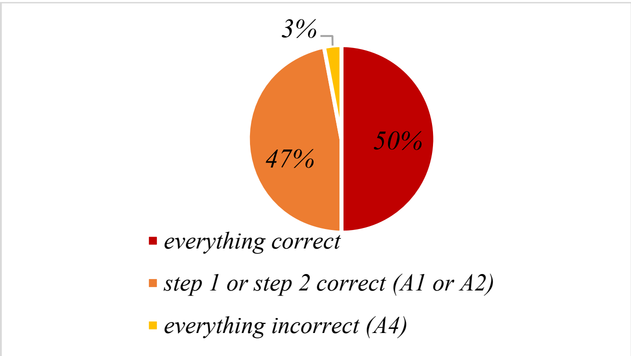
Graph 6. Implicative graph from the solutions of problems

Evaluating and Analysis of Problem Difficulty

The pupils managed problem A very well. Half of the pupils solved it 100%. One pupil could not solve the problem at all. The remaining pupils were mostly wrong in that they did not notice the two axes of symmetry, but only the first one. Therefore, they displayed all the notes only according to the axis o_1 (this was the most frequent error in the pupils' solutions). The other pupils (sixteen pupils) had no problem in understanding and solving this problem, so the solution was very easy for them. Also twelve pupils had a problem understanding the problem and six pupils of them stated that the problem was very difficult. More analysis can be seen in the Graph 7. and Graph 8.



Graph 7. Evaluating the difficulty of the problem A



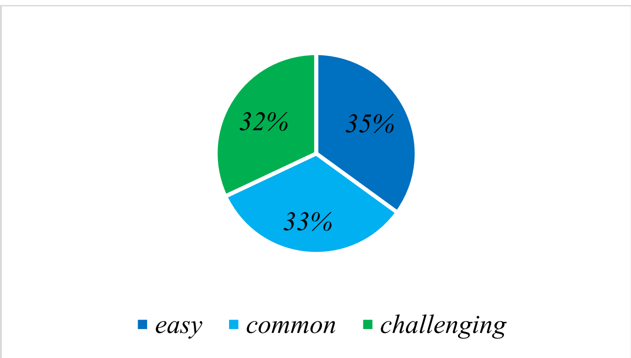
Graph 8. Evaluating the success of the problem A

Problem B seemed to be a bit confusing for some pupils. They did not find it easy to choose from among the notes those that were symmetrical along the given axes. It required precise drawing and measuring, and this, according to our findings, many pupils did not want to do.

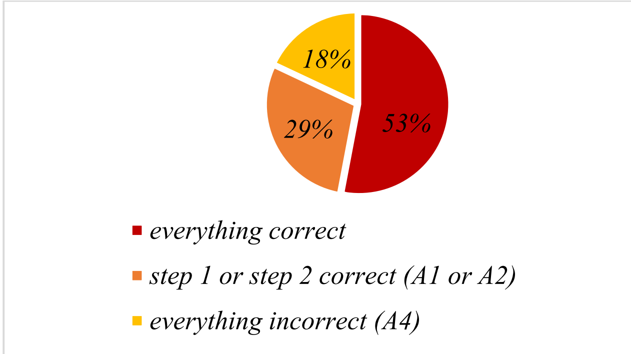
We present one correct solution in Figure 7.

Figure 7. The pupil solution

In the end, however, more than half of the pupils successfully solved problem B and only six pupils did not solve it correctly. Opinions on the difficulty of the problem B varied (see Graph 9. and Graph 10.).



Graph 9. Evaluating the difficulty of the problem B



Graph 10. Evaluating the success of the problem B

We offered the pupils that they could find and mark another solution to the problem or another symmetry. Out of all the pupils, only one pupil to do so. In her solution, she determined the center of symmetry, also marked the pattern and the image of this symmetry. The Figure 8. presents a particular creative solution of the mentioned pupil.

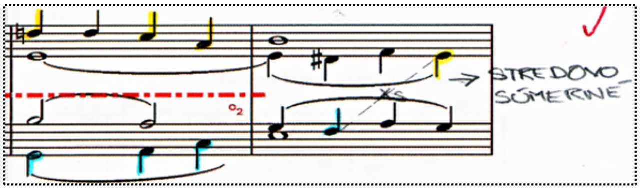
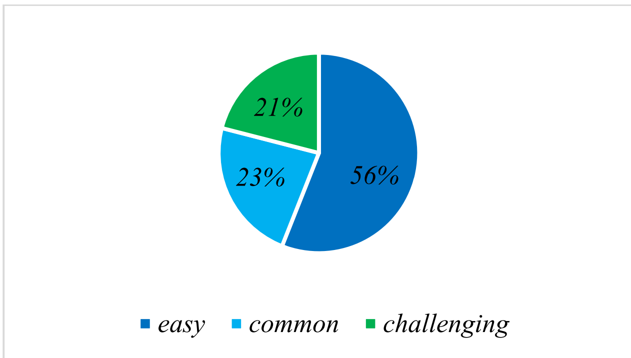
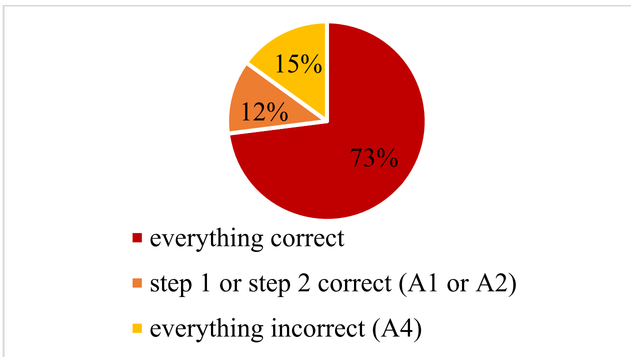


Figure 8. Part of the creative solution to problem B

Problem C was well liked by two-thirds of the pupils and did not present anything exceptionally challenging for them. The results we found are shown in the Graph 11. and Graph 12.



Graph 11. Evaluating the difficulty of the problem C



Graph 12. Evaluating the success of the problem C

Our analysis confirms the results of their work: twenty-three pupils solved this task perfectly. We learned from the direct reactions of some pupils that they did not feel like drawing. However, when correcting the worksheets, we were surprised by the success rate of this task. One pupil wrote: "The problem C was more difficult in that I didn't know what a violin bridge was, plus it is difficult to transfer a picture with irregular sides." Ironically, this pupil received full marks for this task. Another pupil wrote us a message for the task: "I am an artist and artists are not good at math." We can see that despite achieving poor results in mathematics lessons, this pupil's creativity and talent were very much in evidence. Thus, this pupil also received full points for solving the problem. One pupil, in addition to completing the basic image of a violin bridge, also created an image of a shadow that was created when taking a photograph.

The following actual pupil solutions (see Figure 9.) show that the pupils worked conscientiously on the task and completed their concrete solution by drawing, shading, filling in the background or using color. Shading of the so-called violin bridge is visible on the top right.

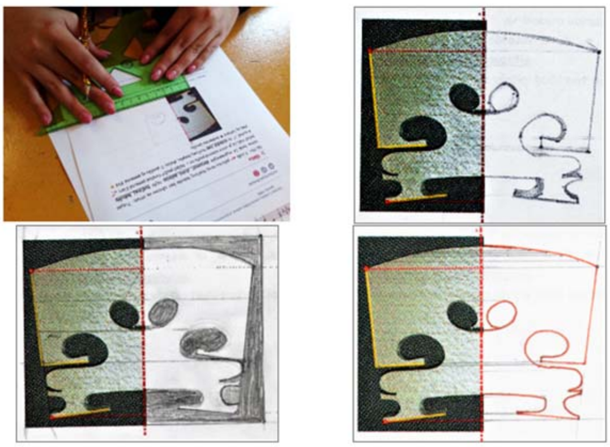


Figure 9. Various solutions to problem C by pupils

4. Conclusion

Mathematics is a link between individual scientific subjects' disciplines, through it we can prove or disprove various statements. It teaches how to be systematic, pupils acquire the ability to think systematically and effectively, it leads pupils to independent thinking and creative problem solving. Even though non-standard problems are not included in the teaching process very often, pupils like them and according to our research, they have no problem solving them. We can state that such problems arouse interest in geometry and broaden pupils' general overview. Therefore, it is appropriate and even necessary that, to the greatest possible extent, problems of this type are also introduced into the curriculum of secondary schools. Our findings also confirm the authors in [7], [8].

For the interaction of mathematical terms and music terms we can use sentences said by a well-known mathematician and at the same time a talented pianist and organist Dr. h. c. prof. RNDr. Beloslav Riečan, DrSc. [9]: "Math and music are independent. There are people without musical hearing, called musically deaf people, to be unable to sing a song. I can also imagine mathematically deaf people. However, I think music is a kind of art and art has a role in human life. And a mathematician is also human. Conversely, we tend to talk phrase that mathematics teaches you to think correctly. But one can also think well without mathematical formulas."

Acknowledgements

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