Gridpoint Method for Proving Combinatorial Identities

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Abstract **– Proofs are an important part of mathematical understanding. The three basic methods of proving combinatorial identities are mathematical induction, algebraic calculation, and combinatorial proofs. The last two of them are usually based on so-called double counting, which means counting the number of elements in one group with two different methods. In this article, we show an approach that uses gridpoints (points with integer coordinates) to calculate the number of elements of a set expressed by the left and the right side of a combinatorial identity. The gridpoint method for combinatorial calculation is known from [1] and [2]. This article presents the advantages of gridpoint approach in two aspects. The first one is simplifying the proofs in some cases, and the second one, to show students a way for independent work in invention (or reinvention) of combinatorial identities using the gridpoint method combining integer coordinates in an appropriate way. Finally, we discuss the acceptance of proofs by the gridpoint method as explanatory proofs.**

Keywords **– mathematical education, combinatorial identities, explanatory proof, generalisation, gridpoint method.**

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1. Introduction

Mathematical proofs play an important role not only in the logical structure of mathematics itself as an abstract subject but also in teaching and understanding the essentials of mathematics more deeply. Proofs as examples of exhaustive deductive reasoning are integral parts of mathematics that many times show a way for exploring and inventing the given statement. Evidently, exploration and proving are different activities but they are corresponding, and they enhance each other. They are both fundamental in rigorous mathematical understanding. "Exploration leads to discovery, while proof is a confirmation" [3] (p. 14). The purpose of the proof is validation of the correctness of a statement, however, in mathematical education at the same time, the proof is contributing to knowledge construction. According to [4] (p. 197), "a general consensus has been achieved on the fact that cultivating the sense of proof is an essential part of mathematical education and seems to be an emerging trend to include mathematical proofs into the curriculum". Educational studies [5], [6], [7] show a close interconnection between investigation, generalisation, proving, and combinatorial competencies.

Combinatorial identities are part of the curriculum in mathematics teaching education for discrete mathematics. There are very different methods for proving such kind of formulas. In each case, a new proof sheds new light on the identity, offering deeper understanding from a different viewpoint. A nice example for this is the series of articles [8-15], where the reader can follow different viewpoints to a combinatorial identity [8], [9], 10], generalisation [11], various proofs via generating functions [10], with the help of Jensen's formula [12], using Cauchy's integral formula [14] and others. Article [13] provides a combinatorial proof based on suitable configurations involving dominos and colourings with possibility of generalisation.

Including combinatorics into the standard mathematics curriculum was the idea based on the expectation that it would help students improve their problem-solving skills [16]. In teacher training, we try to offer the most efficient and less complicated methods that are available for students and serve also as a motivation to their next study. If the proof does not help acquire deeper knowledge, it remains meaningless in the eyes of students [4].

Two of the most usual methods for proving combinatorial identities are mathematical induction and counting arguments (double counting). For the second one, we identify a set counted by the formulas in the identity and provide two counting arguments to explain each side of the identity. We can do the counting argument mentioned above by looking at the set as gridpoints. That is, the elements counted by the formula can be seen as gridpoints with integer coordinates. By gridpoints we refer to tuples (x_1, x_2), with $x_1, x_2 \in \mathbb{Z}$ in the plane, or generally, to m-uples $(x_1, x_2, \dots, x_m)$, $x_i \in \mathbb{Z}$, $i = 1, 2, \dots, m$, in the m-dimensional space.

The didactical advantage of using gridpoint method in teaching combinatorial identities is in the broader possibility of investigation and generalization. We do not need to know a given combinatorial identity in advance (as obviously we need it when using other proving methods); using this method, we can construct different point sets of gridpoints and reinvent/rediscover or possibly invent/discover a combinatorial identity [1, 2]. In the next sessions, we present both options.

2. Mathematical Background

The most usual way of proving combinatorial identities is mathematical induction, e.g., in [17] it is the prevailing method for proving basic identities. For illustration, here we show the proof of Vandermonde´s identity (also called Cauchy´s combinatorics formula or additive theorem for binomial coefficients [17]).

Theorem 1. *For any positive integers u, v and any non-negative integer m, such that* $m \leq u + v$,

$$\sum_{k=0}^{m} \binom{u}{k}\binom{v}{m-k} = \binom{u+v}{m}. \tag{1}$$

Inductive proof of Theorem 1. We prove the identity with mathematical induction. For $m = 0$ we have identity 1 = 1. Now we suppose that identity (1) is valid for some non-negative m and prove it for $m + 1$. It means proving that the identity

$$\sum_{k=0}^{m+1} \binom{u}{k}\binom{v}{m+1-k} = \binom{u+v}{m+1} \tag{2}$$

follows from (1). Using the algebraic definition of binomial numbers, we have that

$$\binom{u+v}{m+1} = \frac{(u+v)!}{(m+1)!\,(u+v-m-1)!} = \\ = \frac{u+v-m}{m+1}\binom{u+v}{m}.$$

Therefore, to prove the identity (2) it is enough to show that multiplying the left side of (1) by the same fraction (i.e. (u+v-m)/(m+1)) we get the left side of (2). First, we rewrite this fraction as follows.

$$\frac{u+v-m}{m+1} = \frac{1}{m+1}\left(v-m+k+(u-k)\right) = \\ = \frac{1}{m+1}\left[\frac{(m-k+1)(v-m+k)}{m-k+1} + \frac{(k+1)(u-k)}{k+1}\right].$$

Now we calculate

$$\frac{u+v-m}{m+1}\sum_{k=0}^{m}\binom{u}{k}\binom{v}{m-k} = \\ = \frac{1}{m+1}\left[\sum_{k=0}^{m}(m-k+1)\binom{u}{k}\binom{v}{m-k}\frac{v-m+k}{m-k+1} + \sum_{k=0}^{m}(k+1)\binom{u}{k}\frac{u-k}{k+1}\binom{v}{m-k}\right] = \\ = \frac{1}{m+1}\left[\sum_{k=0}^{m}(m-k+1)\binom{u}{k}\binom{v}{m+1-k} + \sum_{k=0}^{m}(k+1)\binom{u}{k+1}\binom{v}{m-k}\right] = \\ = \frac{1}{m+1}\left[(m+1)\binom{u}{0}\binom{v}{m+1} + \sum_{k=1}^{m}(m-k+1)\binom{u}{k}\binom{v}{m+1-k} + \sum_{k=0}^{m-1}(k+1)\binom{u}{k+1}\binom{v}{m-k} + (m+1)\binom{u}{m+1}\binom{v}{0}\right].$$

Rewriting $k + 1$ to k in the second sum (by substitution $k' = k + 1$), we have

$$\sum_{k=1}^{m}(m-k+1)\binom{u}{k}\binom{v}{m+1-k} + \sum_{k=1}^{m} k\binom{u}{k}\binom{v}{m+1-k} = (m+1)\sum_{k=1}^{m}\binom{u}{k}\binom{v}{m+1-k}.$$

So, we can summarise

$$\frac{u+v-m}{m+1}\sum_{k=0}^{m}\binom{u}{k}\binom{v}{m-k} =$$
$$= \frac{1}{m+1}\Bigg[(m+1)\binom{u}{0}\binom{v}{m+1} + (m+1)\sum_{k=1}^{m}\binom{u}{k}\binom{v}{m+1-k} + (m+1)\binom{u}{m+1}\binom{v}{0}\Bigg] =$$
$$= \sum_{k=0}^{m+1}\binom{u}{k}\binom{v}{m+1-k}.$$

We got the left-hand side of equation (2) and with that the proof is complete.

3. Results

In this section, we introduce the gridpoint method and use it for proving some basic combinatorial identities.

The method is based on the creation of a set of gridpoints so that the number of elements of this set is calculated in two different ways. The two ways represent the two sides of the given identity to prove. The gridpoints are represented by their coordinates which are appropriate integers.

3.1. Gridpoints with Coordinates 0 and 1

Let us illustrate the gridpoint method by proving Vandermonde´s identity (1) using it. In this case, we use gridpoints *x* with coordinates

$x = (x_1, x_2, \dots, x_u, x_{u+1}, \dots, x_{u+v})$, where x_i equals to 0 or 1 (for $i = 1, 2, \dots, u+v$).

Proof 2 of Theorem 1. We create a set of gridpoints G in which there are 1s precisely on *n* places and elsewhere there are zeros. The number of elements of this set is $\binom{u+v}{n}$.

Now, we calculate the number of elements of G in a different way. We consider each gridpoint from G as a point with coordinates $x = ((x_1, x_2, \dots, x_u), (x_{u+1}, \dots, x_{u+v}))$ where the number of 1s in $(x_1, x_2, \dots, x_u)$ is *k*, and the number of 1s in $(x_{u+1}, \dots, x_{u+v})$ is *m-k*. Therefore, we can express the number of elements of G as

$$\sum_{k=0}^{m}\binom{u}{k}\binom{v}{m-k}.$$

We have calculated the cardinality of G in two ways, so it means that

$$\binom{u+v}{m} = \sum_{k=0}^{m}\binom{u}{k}\binom{v}{m-k}.$$

Note: A special case of Vandermonde´s identity for $u = v = m$ is $\binom{2m}{m} = \sum_{k=0}^{m}\binom{m}{k}^2$. This special combinatorial identity was proved in [18] using Cauchy´s integral formula from the theory of complex variable functions. Different and much simpler proof (called Hat Proof) is shown in [19]: We have 2*m* different hats. We divide them into two groups, each with *m* hats. Then we choose *k* hats from the first group and *m – k* hats from the second one. There are $\binom{m}{k}^2$ ways to do this. Evidently, to create all possible choices of *m* hats from the 2*m* hats, we have to choose $k = 0, 1, \dots, m$ hats from the first *m* hats, and the remaining *m - k* hats from the second *m* hats. Then the sum over all such k is the number of possibilities to choose m hats from $2m$ hats. Therefore, $\sum_{k=0}^{m}\binom{m}{k}^2 = \binom{2m}{m}$, as desired [19].

Theorem 2. *For any natural numbers m and k*

$$\binom{m}{0} + \binom{m+1}{1} + \dots + \binom{m+k}{k} = \binom{m+k+1}{k}. \quad (3)$$

Proof of Theorem 2. Consider the gridpoints $x = (x_1, x_2, \dots, x_m, x_{m+1}, \dots, x_{m+k+1})$, where $x_i = 0$ or 1 for $i = 1, 2, \dots, m+k+1$. Denote by G the set of gridpoints *x* such that there are exactly k 1s in the sequence of coordinates $x_1, x_2, \dots, x_{m+k+1}$ in total. Then the number of elements of G is $|G| = \binom{m+k+1}{k}$, which is the right side of (3).

Now we calculate the cardinality of G in a different way by decomposing G into disjoint sets G_j, $j = 0, 1, \dots, k$ depending on the number of 1s at the end of the sequence. That is, G_j denotes the set of gridpoints $x = (x_1, x_2, \dots, x_{m+k+1})$ such that $x_{m+k+1} = x_{m+k} = \dots = x_{m+k-j+2} = 1$ and $x_{m+k-j+3} = 0$. G_j are disjoint sets for different j, $j = 0, 1, \dots, k$, and $G = \bigcup_{j=0}^{k} G_j$. For $j = 0$, set G_0 contains such gridpoints $x = (x_1, x_2, \dots, x_{m+k+1})$ that $x_{m+k+1} = 0$, and on the remaining $m + k$ places are *k* number of 1s. Therefore

$$|G_0| = \binom{m+k}{k}.$$

For $j = 1$, we have $x_{m+k+1} = 1$, $x_{m+k} = 0$, and on the remaining $(m+k-1)$ places (in the sequence of coordinates $x_1, x_2, \dots, x_{m+k-1}$) are exactly $(k-1)$ 1s. It means that

$$|G_1| = \binom{m+k-1}{k-1}.$$

For $j = k-1$ the set G_{k-1} contains such gridpoints $x = (x_1, x_2, \dots, x_{m+k+1})$ that $x_{m+k+1} = x_{m+k} = \dots = x_{m+3} = 1$, $x_{m+2} = 0$, and one of the remaining $m + 1$ places is equal to 1.

$$|G_{k-1}| = \binom{m+1}{1}$$

For $j = k$ the set G_k is a single element set with element $x = (0, \dots, 0, 1, \dots, 1)$, where $x_{m+k+1} = x_{m+k} = \dots = x_{m+2} = 1$, and $x_{m+1} = x_m = \dots = x_1 = 0$.

$$|G_k| = 1 = \binom{m}{0}$$

The cardinality of set G is equal to the sum of cardinalities of its disjoint parts:

$$|G| = \binom{m+k+1}{k} = \binom{m}{0} + \binom{m+1}{1} + \dots + \binom{m+k}{k} = \sum_{j=0}^{k} \binom{n+j}{j},$$

and the combinatorial identity (3) is proven.

3.2. Gridpoints with Various Coordinate Values

Now, we introduce a proof of a combinatorial identity using gridpoints with different coordinate values, not only 0s and 1s.

Theorem 3. *For all natural numbers m and k*

$$\sum_{k=0}^{m} k\binom{m}{k} = m2^{m-1}.$$

Proof of Theorem 3. Consider the set G of gridpoints $x = (x_1, x_2, \dots, x_m)$ such that $x_1 \in \{1, 2, \dots, m\}$ and $x_j \in \{0,1\}$ for $j = 2, 3, \dots, m$. Then $|G| = m.2^{m-1}$ since there are *m* choices for x_1 and 2 choices for x_j for $j = 2, 3, \dots, m$.

Calculating $|G|$ differently, we suppose G is a union of sets Gj, where $j = 0, 1, \dots, m-1$. Each Gj is a set of such gridpoints that have among $x_2, x_3, \dots, x_m$ precisely *j* number of 1s. Then

$$|G_j| = m\binom{m-1}{j}. \quad (4)$$

Now we use substitution $j = k - 1$ in (4). Since $0 \le k - 1 \le m - 1$, we have $1 \le k \le m$. Then

$$m\binom{m-1}{j} = m\frac{(m-1)!}{(m-1-j)!\,j!} = \frac{m!}{(m-k)!\,(k-1)!} = k\frac{m!}{(m-k)!\,k!} = k\binom{m}{k}.$$

Calculating $|G|$, we get the desired combinatorial identity

$$|G| = m.2^{m-1} = \sum_{j=0}^{m-1} m\binom{m-1}{j} = \sum_{k=0}^{m} k\binom{m}{k}.$$

In the previous proof, coordinate x_1 of gridpoint *x* could be different from 0 or 1. The next identity, presented in Theorem 4, follows from considering the same gridpoint problem when other coordinates (not only x_1) could have values different from 0 and 1. The process of creating a sequence of such gridpoints is shown in Proof 1 of the theorem.

Theorem 4. *For any non-negative integer m and any constant non-negative integer l such that* $0 \le l \le m$,

$$\sum_{k=l}^{m} \binom{m}{k}\binom{k}{l} = \binom{m}{l} 2^{m-l}.$$

Proof 1 of Theorem 4. First, we prove the identity

$$\sum_{k=l}^{m} k(k-1)(k-2)\dots(k-l+1)\binom{m}{k} = m(m-1)\dots(m-l+1)2^{m-l} \quad (5)$$

where $l \le m$ is constant. Consider the set M of gridpoints $x = (x_1, x_2, \dots, x_m)$ such that $x_j \in \{1, 2, \dots, m-j+1\}$ for $j = 1, 2, \dots, l$ and $x_j \in \{0,1\}$ for $j = l+1, \dots, m$.

Calculating the number of elements of set *M*, we obtain

$$|M| = m(m-1)\dots(m-l+1)2^{m-l},$$

which is the right-hand side of the expression (5).

Now we calculate the number of gridpoints the other way. We create sets M_j $(0 \le j \le n-l)$ of such gridpoints $x \in M$ that the number of 1s among their last $n-l$ coordinates $x_{l+1}, x_{l+2}, \dots, x_n$ is exactly *j*. The number of elements of such sets is:

$$|M_0| = m(m-1)\dots(m-l+1)$$

$$|M_1| = m(m-1)\dots(m-l+1)\binom{m-l}{1}$$

$$|M_2| = m(m-1)\dots(m-l+1)\binom{m-l}{2}$$

$$\dots$$

$$|M_{m-l}| = m(m-1)\dots(m-l+1)\binom{m-l}{m-l}$$

The number of elements in set *M* is the sum for all *j*, $0 \le j \le m-l$.

$$|M| = \sum_{j=0}^{m-l} |M_j| = \sum_{j=0}^{m-l} m(m-1)\dots(m-l+1)\binom{m-l}{j}.$$

Substituting

$$\binom{m-l}{j} = \frac{(m-l)(m-l-1)\dots(m-l-j+2)(m-l-j+1)}{j!},$$

reorganising the product and using the symmetry of the binomial coefficients we obtain

$$|M| = \sum_{j=0}^{m-l} \frac{m(m-1)\dots(m-l+1)(m-l)\dots(m-l-j+1)}{j!}$$

$$= \sum_{j=0}^{m-l} \binom{m}{j}(m-l-j+l)(m-l-j+l-1)\dots(m-l-j+2)(m-l-j+1) =$$

$$= \sum_{j=0}^{m-l} \binom{m}{m-j}(m-j)(m-j-1)\dots(m-j-l+2)(m-j-l+1).$$

By substitution $k = m - j$ we obtain

$$|M| = \sum_{k=l}^{m} \binom{m}{k} k(k-1) \dots (k-l+2)(k-l+1).$$

So, we have proved the identity (5). Dividing both sides of (5) by $l!$ we get the desired combinatorial identity.

$$\sum_{k=l}^{m} \binom{m}{k} \frac{k(k-1) \dots (k-l+1)}{l!} = \frac{m(m-1) \dots (m-l+1)}{l!} 2^{m-l}$$

$$\sum_{k=l}^{m} \binom{m}{k}\binom{k}{l} = \binom{m}{l} 2^{m-l}.$$

Next, we present another proof of Theorem 4 that also uses the gridpoint method.

Proof 2 of Theorem 4. First, we prove the combinatorial identity

$$\sum_{j=0}^{m} \binom{m}{j}\binom{m-j}{l} = \binom{m}{l} 2^{m-l}. \quad (6)$$

We consider a set M of such gridpoints $x = (x_1, x_2, \dots, x_m)$, that they have zero coordinates on exactly l places and all other coordinates (on the remaining $n - l$ places) are equal to 1 or 2. According to this definition, the number of elements of set M is

$$|M| = \binom{m}{l} 2^{m-l}.$$

Now, we partition the set M into subsets M_j $(0 \le j \le m - l)$ that the j-th set contains gridpoints that have exactly j coordinates equal to 1. Thus, the elements in the set M_j have zeros on l coordinates from $m - j$ places and the remaining coordinates are equal to 2. Therefore

$$|M_j| = \binom{m}{j}\binom{m-j}{l}, \text{ and}$$

$$|M| = \sum_{j=0}^{m-l} |M_j| = \sum_{j=0}^{m-l} \binom{m}{j}\binom{m-j}{l},$$

which is the left-hand side of identity (6). By substitution $j = k - l$ we have

$$|M| = \sum_{k=l}^{m} \binom{m}{k-l}\binom{m-k+l}{l},$$

where $l \le k \le m$.

Since $\binom{m}{k-l}\binom{m-k+l}{l} = \frac{m!}{(k-l)!\,(m-k+l)!} \frac{(m-k+l)!}{l!(m-k)!} = \frac{m!}{k!\,(m-k)!} \frac{k!}{l!(k-l)!} = \binom{m}{k}\binom{k}{l}$, we have proved that

$$\sum_{k=l}^{m} \binom{m}{k}\binom{k}{l} = \binom{m}{l} 2^{m-l}.$$

4. Conclusion

The aim of this article was twofold: to introduce a less usual method for proving combinatorial identities and to demonstrate the usefulness of the gridpoint method for further investigation in the field of combinatorial identities.

Different types of proofs are perceived differently by undergraduate mathematics students. From the perspective of students, the most meaningful proofs are the explanatory ones, which bring extra sight into the problem. Lockwood [20] realised that from the student´s viewpoint, a proof can be judged as less or more explanatory according to the student´s individual reading. Generally saying, proofs with complicated algebraic demonstrations are less frequently perceived as explanatory while proofs with possible visual reasoning are more often judged as explanatory [20].

In the field of combinatorics, most proofs fall into three categories:

- Proofs with mathematical induction
- Proofs with algebraic calculation
- Combinatorial proofs.

According to [21], correct combinatorial proofs may be considered to be explanatory proofs, "however, combinatorial proofs also vary from other types of proof, such as induction or algebraic proofs, in several important ways" [21] (p. 2). "One of characteristics of combinatorial proofs is that they consist entirely of sentences and paragraphs, which verbally explain the symbols that are used in the identity without algebraic manipulation of those symbols. This characteristic could have possible connotations for students, as some students have been found to be less probably to admit an argument as an accurate mathematical proof if it does not include symbolic mathematical adjustments" [21] (p. 2). In this comparison, the gridpoint method can be characterised as a method that uses less algebraic calculation and offers a more rigorous mathematical context as a combinatorial proof. Therefore, it means that the gridpoint method of proving may have a stronger acceptance than the combinatorial proof and algebraic proof from the students' side as real explanatory proof.

The second advantage of this method inheres to the possibility of its variability. By constructing various appropriate sets of gridpoints one can discover new combinatorial identities as it was shown also in proofs of Theorem 4.

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