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Existence and Local Stability of Prime Period-two Solutions of Certain Quadratic Rational Second Order Difference Equation

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Abstract **– In this paper we proved the existence and local stability of prime period-two solutions for the equation** $x_{n+1}=\frac{\alpha x_n^2+\beta x_n+\gamma x_{n-1}}{Ax_n^2+Bx_n+Cx_{n-1}}$**, for certain values of parameters α,β,γ,A,B,C≥0, where α+β+γ>0 , A+B+C>0, and where the initial conditions** $x_{-1}, x_0>0$ **are arbitrary real numbers such that at least one is strictly positive. For the obtained periodic solutions, it is possible to be locally asymptotically stable, saddle points or non-hyperbolic points. The existence of repeller points is not possible.**

Keywords **- bifurcation, difference equation, equilibrium, local stability, prime period-two.**

1. Introduction

In this paper, we will look at the existence and local stability of the equation

$$x_{n+1}=\frac{\alpha x_n^2+\beta x_n+\gamma x_{n-1}}{Ax_n^2+Bx_n+Cx_{n-1}}, \quad (1)$$

where $\alpha,\beta,\gamma,A,B,C\geq 0$, and $\alpha+\beta+\gamma>0$, $A+B+C>0$.

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Initial conditions x_{-1}, x_0 are non-negative real numbers and at least one of them is strictly positive. Rational differential equations with quadratic terms were first significantly observed in papers [1] and [2]. Rational equations with quadratic terms are studied in more detail in the papers [3], [4] and [5]. Similar results obtained in [3] were obtained for another rational equation with quadratic terms, but in which the repeller equilibrium point occurs, see [6]. In both of these papers, the results of prime period-two solutions were used to obtain results on the global stability of the observed equations. The existence and uniqueness of the positive equilibrium of Eq. (1) has been proven before and will be omitted here. Detailed analysis is dedicated to the existence and stability of prime period-two solutions for some special non-negative parameter values.

2. Existence of Prime Period-two Solutions

Suppose that {φ,ψ} is a prime period-two solution of Eq.(1). Then it is

$\varphi=f(\psi,\varphi)$ and $\psi=f(\varphi,\psi)$, where $\varphi,\psi\in[0,\infty)$ and $\varphi\neq\psi$.

This is equivalent to the following system of equations

$$\varphi=\frac{\alpha\psi^2+\beta\psi+\gamma\varphi}{A\psi^2+B\psi+C\varphi}$$
$$\psi=\frac{\alpha\varphi^2+\beta\varphi+\gamma\psi}{A\varphi^2+B\varphi+C\psi}.$$

where φ≠ψ. We can write the last system in form

$$\varphi(A\psi^2+B\psi+C\varphi)=\alpha\psi^2+\beta\psi+\gamma\varphi \quad (2)$$
$$\psi(A\varphi^2+B\varphi+C\psi)=\alpha\varphi^2+\beta\varphi+\gamma\psi \quad (3)$$

where φ≠ψ.

Theorem 1: Equation (1) has a prime period-two solution $\{0,\frac{\gamma}{C}\}$ if $\alpha=\beta=0$. There are no other prime period-two solutions for which it is $\varphi\psi=0$.

Proof: If in the equations (2) and (3) we put that $\varphi=0$, it implies that

$$0 = \alpha\psi^2 + \beta\psi \text{ and } C\psi^2 = \gamma\psi,$$

which is true if and if it is $\alpha = \beta = 0$ and $\psi = \frac{\gamma}{C}$. From the last system it is not difficult to see that there are no other prime period-two solutions for which it is $\varphi\psi = 0$.

We will now determine the other prime period-two solutions for which it is $\varphi\psi \neq 0$.

If we subtract the equations (2) and 3) then we have

$$(\psi - \varphi)(A\varphi\psi - C(\varphi + \psi) - \alpha(\varphi + \psi) - \beta + \gamma) = 0. \quad (4)$$

If we divide equation (2) by φ and divide equation (3) by ψ, and then we subtract the equations thus obtained then we get

$$(\psi - \varphi)\left(A(\varphi + \psi) + B - C - \alpha\left(\frac{(\varphi+\psi)^2 - \varphi\psi}{\varphi\psi}\right) - \beta\left(\frac{\varphi+\psi}{\varphi\psi}\right)\right) = 0. \quad (5)$$

If we put

$$\varphi + \psi = x \text{ and } \varphi\psi = y,$$

where is $x > 0, y > 0$ and $x^2 - 4y > 0$, then φ and ψ are positive and different solutions of the quadratic equation

$$t^2 - xt + y = 0. \quad (6)$$

Based on new labels equations (4) and (5) we can write in the form of a system

$$\begin{cases} Ay - (C + \alpha)x - \beta + \gamma = 0 \\ Axy - \alpha x^2 + (B - C)y + \alpha y - \beta x = 0. \end{cases} \quad (7)$$

Theorem 2: For Eq.(1) the following holds:

a) If $\alpha = \beta = 0$, then Eq.(1) has two prime period-two solutions

$$\{\varphi_1,\psi_1\}=\{0,\tfrac{\gamma}{C}\}$$

and $\{\varphi_2,\psi_2\}$, where

$$\varphi_2 = \frac{C - B - \sqrt{(B - C)(B + 3C) + 4A\gamma}}{2A}$$
$$\psi_2 = \frac{C - B + \sqrt{(B - C)(B + 3C) + 4A\gamma}}{2A}$$

if and only if

$$C > B \text{ and } \frac{(C-B)(B+3C)}{4\gamma} < A < \frac{C(C-B)}{\gamma}.$$

b) If $C = \alpha = 0$, then Eq.(1) has prime period-two solution

$$\varphi = \frac{B(\beta - \gamma) - \sqrt{(\beta - \gamma)(B^2(\beta - \gamma) - 4A\gamma^2)}}{2A\gamma}$$
$$\psi = \frac{B(\beta - \gamma) + \sqrt{(\beta - \gamma)(B^2(\beta - \gamma) - 4A\gamma^2)}}{2A\gamma}$$

if and only if

$$\beta > \gamma \text{ and } B^2(\beta - \gamma) > 4A\gamma^2.$$

Proof:

a) Suppose that $\alpha = \beta = 0$. System (7) is now in the form

$$\begin{cases} Ay - Cx + \gamma = 0 \\ Axy + (B - C)y = 0. \end{cases}$$

The solutions of the last system are

$$x_1 = \frac{\gamma}{C} \text{ and } y_1 = 0,$$
$$x_2 = \frac{C-B}{A} \text{ and } y_2 = \frac{C(C-B)-A\gamma}{A^2}$$

where is $x_2 > 0$, $y_2 > 0$ and ${x_2}^2 - 4y_2 > 0$ if and only if

$$C > B \text{ and } \frac{(C-B)(B+3C)}{4\gamma} < A < \frac{C(C-B)}{\gamma},$$

Since

$$x_2^2 - 4y_2 = \frac{(B - C)(B + 3C) + 4A\gamma}{A^2}.$$

From $x_1 = \frac{\gamma}{C}$ and $y_1 = 0$ we conclude that one prime period-two solution is $\{\varphi,\psi\}=\{0, \frac{\gamma}{C}\}$. Now, the solution of the equation

$$t^2 - \left(\frac{C - B}{A}\right)t + \frac{C(C - B) - A\gamma}{A^2} = 0$$

are

$$t_{\pm} = \frac{C - B \pm \sqrt{(B - C)(B + 3C) + 4A\gamma}}{2A}.$$

Hence, we conclude that the second prime period-two solution is

$$\varphi_2 = \frac{C - B - \sqrt{(B - C)(B + 3C) + 4A\gamma}}{2A}$$
$$\psi_2 = \frac{C - B + \sqrt{(B - C)(B + 3C) + 4A\gamma}}{2A}.$$

b) Suppose that $C = \alpha = 0$. System (7) is now in the form

$$\begin{cases} Ay - \beta + \gamma = 0 \\ Axy - y - \beta x = 0, \end{cases}$$

whose solutions are

$$x = \frac{B(\beta - \gamma)}{A\gamma} \text{ and } y = \frac{\beta - \gamma}{A}.$$

Since

$$x^2 - 4y = \frac{(\beta - \gamma)(B^2(\beta - \gamma) - 4A\gamma^2)}{A^2\gamma^2}$$

it is $x > 0, y > 0$ and $x^2 - 4y > 0$ if and only if

$$\beta > \gamma \text{ and } B^2(\beta - \gamma) > 4A\gamma^2.$$

Now it is not difficult to see that the solutions of the equation

$$t^2 - \left(\frac{B(\beta-\gamma)}{A\gamma}\right)t + \frac{\beta-\gamma}{A} = 0$$

are

$$t_\pm = \frac{B(\beta-\gamma) \pm \sqrt{(\beta-\gamma)(B^2(\beta-\gamma)-4A\gamma^2)}}{2A\gamma}.$$

So, the prime period-two solution is

$$\varphi = \frac{B(\beta-\gamma) - \sqrt{(\beta-\gamma)(B^2(\beta-\gamma)-4A\gamma^2)}}{2A\gamma}$$

$$\psi = \frac{B(\beta-\gamma) + \sqrt{(\beta-\gamma)(B^2(\beta-\gamma)-4A\gamma^2)}}{2A\gamma}.$$

Theorem 3: For Eq.(1) the following holds:

a) If $\gamma = 0$, then Eq.(1) has no prime period-two solution.

b) If $A = 0$, then Eq.(1) has prime period-two solution

$$\varphi = \frac{1}{2}\left(\frac{\gamma-\beta}{C+\alpha} - \sqrt{\frac{(\gamma-\beta)(B(\gamma-\beta)-C(3\beta+\gamma)-\alpha(\beta+3\gamma))}{(B-C+\alpha)(C+\alpha)^2}}\right)$$

$$\psi = \frac{1}{2}\left(\frac{\gamma-\beta}{C+\alpha} + \sqrt{\frac{(\gamma-\beta)(B(\gamma-\beta)-C(3\beta+\gamma)-\alpha(\beta+3\gamma))}{(B-C+\alpha)(C+\alpha)^2}}\right)$$

if and only if

$$B\beta + C(3\beta+\gamma) + \alpha(\beta+3\gamma) < B\gamma.$$

Proof :

a) Suppose that $\gamma = 0$. Then system (1) has the form

$$\begin{cases} Ay - (C+\alpha)x - \beta = 0 \\ Axy - \alpha x^2 + (B-C)y + \alpha y - \beta x = 0, \end{cases}$$

which solutions are

$$x_\pm = \frac{-(B-C+\alpha) \pm \sqrt{(B-C+\alpha)^2(C+\alpha)^2 + 4AC\beta(-B+C-\alpha)}}{2AC},$$

$$y_\pm = \frac{C^3 + C^2\alpha - B(C+\alpha)^2 - \alpha^3 - C\alpha^2 + 2AC\beta}{2A^2C}$$

$$\pm \frac{(C+\alpha)\sqrt{(B-C+\alpha)^2(C+\alpha)^2 + 4AC\beta(-B+C-\alpha)}}{2A^2C}.$$

As we know it needs to be $x_\pm > 0, y_\pm > 0$ and $x_\pm^2 - 4y_\pm > 0$. However, we have that

$$y_+ = \frac{C^3 + C^2\alpha - B(C+\alpha)^2 - \alpha^3 - C\alpha^2 + 2AC\beta}{2A^2C}$$

$$+ \frac{(C+\alpha)\sqrt{(B-C+\alpha)^2(C+\alpha)^2 + 4AC\beta(-B+C-\alpha)}}{2A^2C}$$

and

$$x_+^2 - 4y_+ = \frac{(B+3C+\alpha)(-C^3 - C^2\alpha + B(C+\alpha)^2 + \alpha^3 + C\alpha^2 - 2AC\beta)}{2A^2C^2}$$

$$- \frac{(B+3C+\alpha)(C+\alpha)\sqrt{(B-C+\alpha)^2(C+\alpha)^2 + 4AC\beta(-B+C-\alpha)}}{2A^2C^2}.$$

The last two relations are of a different sign, from which we conclude that it cannot be $y_+ > 0$ and ${x_+}^2 - 4y_+ > 0$ at the same time. On the other hand, the solution

$$x_- = \frac{-(B-C+\alpha) - \sqrt{(B-C+\alpha)^2(C+\alpha)^2 + 4AC\beta(-B+C-\alpha)}}{2AC}$$

cannot be positive, since if $B - C + \alpha > 0$, then $x_- < 0$, and if $B - C + \alpha < 0$, then

$$(B-C+\alpha)^2(C+\alpha)^2 + 4AC\beta(-B+C-\alpha) - ((B-C+\alpha))^2$$
$$= 4AC\beta(-B+C-\alpha) > 0.$$

So, it is again $x_- < 0$. Based on the above, we conclude that Eq.(1) has no prime period-two solution if $\gamma = 0$.

b) Suppose that $A = 0$. Sistem (1) has a form

$$\begin{cases} -(C+\alpha)x - \beta + \gamma = 0 \\ -\alpha x^2 + (B-C)y + \alpha y - \beta x = 0. \end{cases}$$

The only solution of this system is

$$x = \frac{\gamma-\beta}{C+\alpha} \text{ and } y = \frac{(\gamma-\beta)(C\beta+\alpha\gamma)}{(B-C+\alpha)(C+\alpha)^2}.$$

Since

$$x^2 - 4y = \frac{(\beta-\gamma)(B(\beta-\gamma) + C(3\beta+\gamma) + \alpha(\beta+3\gamma))}{(B-C+\alpha)(C+\alpha)^2}$$

we conclude that $x > 0, y > 0$ and $x^2 - 4y > 0$ if and only if

$$B\beta + C(3\beta+\gamma) + \alpha(\beta+3\gamma) < B\gamma.$$

Positive ond distinct solutions of the quadratic equation

$$t^2 - \left(\frac{\gamma-\beta}{C+\alpha}\right)t + \frac{(\gamma-\beta)(C\beta+\alpha\gamma)}{(B-C+\alpha)(C+\alpha)^2} = 0$$

are

$$t_\pm = \frac{1}{2}\left(\frac{\gamma-\beta}{C+\alpha} \pm \sqrt{\frac{(\gamma-\beta)(B(\gamma-\beta)-C(3\beta+\gamma)-\alpha(\beta+3\gamma))}{(B-C+\alpha)(C+\alpha)^2}}\right).$$

Prime period-two solution is

$$\varphi = \frac{1}{2}\left(\frac{\gamma-\beta}{C+\alpha} - \sqrt{\frac{(\gamma-\beta)(B(\gamma-\beta)-C(3\beta+\gamma)-\alpha(\beta+3\gamma))}{(B-C+\alpha)(C+\alpha)^2}}\right)$$

$$\psi = \frac{1}{2}\left(\frac{\gamma-\beta}{C+\alpha} + \sqrt{\frac{(\gamma-\beta)(B(\gamma-\beta)-C(3\beta+\gamma)-\alpha(\beta+3\gamma))}{(B-C+\alpha)(C+\alpha)^2}}\right).$$

Local Stability of period-two solutions

Set
$u_n = x_{n-1}$ and $v_n = x_n$, for $n = 0,1,2,\ldots$.
Now we can write Eq.(1) in the equivalent form

$$u_{n+1} = v_n$$

$$v_{n+1} = \frac{\alpha v_n^2 + \beta v_n + \gamma u_n}{Av_n^2 + Bv_n + Cu_n}$$

for $n = 0,1,2,\ldots$. Denote with T function defined by

$$T\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}v\\ \dfrac{\alpha v^2 + \beta v + \gamma u}{Av^2 + Bv + Cu}\end{pmatrix}.$$

For a map T defined in this way, the prime period-two solution $\{\varphi, \psi\}$ is a fixed point of map T^2, which is the second iteration of T. We have

$$T^2\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}g(u,v)\\h(u,v)\end{pmatrix} =$$

$$= \begin{pmatrix}\dfrac{\alpha v^2 + \beta v + \gamma u}{Av^2 + Bv + Cu}\\ \dfrac{\alpha\left(\frac{\alpha v^2+\beta v+\gamma u}{Av^2+Bv+Cu}\right)^2 + \beta\left(\frac{\alpha v^2+\beta v+\gamma u}{Av^2+Bv+Cu}\right) + \gamma v}{A\left(\frac{\alpha v^2+\beta v+\gamma u}{Av^2+Bv+Cu}\right)^2 + B\left(\frac{\alpha v^2+\beta v+\gamma u}{Av^2+Bv+Cu}\right) + Cv}\end{pmatrix}.$$

Also, based on the definition

$$J_{T^2}(\varphi,\psi) = \begin{pmatrix}\dfrac{\partial g}{\partial u}(\varphi,\psi) & \dfrac{\partial g}{\partial v}(\varphi,\psi)\\ \dfrac{\partial h}{\partial u}(\varphi,\psi) & \dfrac{\partial h}{\partial v}(\varphi,\psi)\end{pmatrix}.$$

Theorem 4: If $\alpha = \beta = 0$, then Eq.(1) has two prime period-two solutions

$$\{\varphi_1, \psi_1\} = \left\{0, \frac{\gamma}{C}\right\}$$

which is

a) locally asymptotically stable if
$$B \geq C \text{ or } B < C \text{ and } A > \frac{C^2 - BC}{\gamma},$$
b) a saddle point if
$$B < C \text{ and } A < \frac{C^2 - BC}{\gamma},$$
c) a nonhyperbolic point if
$$B < C \text{ and } A = \frac{C^2 - BC}{\gamma}$$

and $\{\varphi_2, \psi_2\}$, where

$$\varphi_2 = \frac{C - B - \sqrt{(B-C)(B+3C) + 4A\gamma}}{2A}$$

$$\psi_2 = \frac{C - B + \sqrt{(B-C)(B+3C) + 4A\gamma}}{2A}$$

which is locally asymptotically stable if and only if

$$C > B \text{ and } \frac{(C-B)(B+3C)}{4\gamma} < A < \frac{C(C-B)}{\gamma}.$$

Proof: The existence of these two periodic solutions was proven earlier. It's not hard to see that it is

$$J_{T^2}\left(0, \frac{\gamma}{C}\right) = \begin{pmatrix}\dfrac{C^2}{BC + A\gamma} & 0\\ -\dfrac{BC}{BC + A\gamma} & 0\end{pmatrix}.$$

It's from here

$$\mathbf{det}J_{T^2}\left(0, \frac{\gamma}{C}\right) = 0$$

and

$$\mathbf{tr}J_{T^2}\left(0, \frac{\gamma}{C}\right) = \frac{C^2}{BC + A\gamma}$$

from which we conclude that

$$\left|\frac{C^2}{BC + A\gamma}\right| < 1$$

if and only if

$$B \geq C \text{ or } B < C \text{ and } A > \frac{C^2 - BC}{\gamma}.$$

It's similar

$$\left|\frac{C^2}{BC + A\gamma}\right| < 1$$

if and only if

$$B < C \text{ and } A < \frac{C^2 - BC}{\gamma}$$

and

$$\left|\frac{C^2}{BC + A\gamma}\right| = 1$$

if and only if

$$B < C \text{ and } A = \frac{C^2 - BC}{\gamma}.$$

For prime period-two solution $\{\varphi_2, \psi_2\}$, where

$$\varphi_2 = \frac{C - B - \sqrt{(B-C)(B+3C) + 4A\gamma}}{2A}$$

$$\psi_2 = \frac{C - B + \sqrt{(B-C)(B+3C) + 4A\gamma}}{2A}$$

we have that

$$\mathbf{det}J_{T^2}(\varphi_2, \psi_2) = \frac{((B-C)C + A\gamma)(-C^2 + A\gamma)}{A^2\gamma^2}$$

and

$$\mathbf{tr}J_{T^2}(\varphi_2, \psi_2) = \frac{6A^2\gamma^2 + C(B^3 + B^2C - 6BC^2 + 4C^3) + A(B^2 + 7BC - 9C^2)\gamma}{A^2\gamma^2}$$

After this, it is

$$|\mathbf{tr}J_{T^2}(\varphi_2, \psi_2)| < 1 + \mathbf{det}J_{T^2}(\varphi_2, \psi_2) < 2$$

if and only if

$$C > B \text{ and } \frac{(C-B)(B+3C)}{4\gamma} < A < \frac{C(C-B)}{\gamma}$$

and this is always when a prime period-two solution $\{\varphi_2, \psi_2\}$ exists.

Theorem 5: If C=α=0, then Eq.(1) has prime period-two solution

$$\varphi = \frac{B(\beta-\gamma) - \sqrt{(\beta-\gamma)(B^2(\beta-\gamma) - 4A\gamma^2)}}{2A\gamma}$$

$$\psi = \frac{B(\beta-\gamma) + \sqrt{(\beta-\gamma)(B^2(\beta-\gamma) - 4A\gamma^2)}}{2A\gamma}$$

which is a saddle point if and only if

$$\beta > \gamma \text{ and } B^2(\beta-\gamma) > 4A\gamma^2.$$

Proof: The existence of this prime period-two solutions was proven earlier. We will now show that this solution is whenever there is a saddle point. Since

$$\mathbf{det} J_{T^2}(\varphi,\psi) = \frac{A\gamma^3}{(\beta-\gamma)(B^2\beta + A(\beta-\gamma)\gamma)}$$

and

$$\mathbf{tr} J_{T^2}(\varphi,\psi) = \frac{B^2(2\beta^2 - 3\beta\gamma + \gamma^2) + A\gamma(\beta^2 - 6\beta\gamma + 6\gamma^2)}{(\beta-\gamma)(B^2\beta + A(\beta-\gamma)\gamma)}$$

we conclude that

$$|\mathbf{tr} J_{T^2}(\varphi,\psi)| > |1 + \mathbf{det} J_{T^2}(\varphi,\psi)|$$

if and only if

$$\beta > \gamma \text{ and } B^2(\beta-\gamma) > 4A\gamma^2.$$

Theorem 6: If $A = 0$, then Eq.(1) has prime period-two solution $\{\varphi, \psi\}$, where

$$\varphi = \frac{1}{2}\left(\frac{\gamma-\beta}{C+\alpha} - \sqrt{\frac{(\gamma-\beta)(B(\gamma-\beta) - C(3\beta+\gamma) - \alpha(\beta+3\gamma))}{(B-C+\alpha)(C+\alpha)^2}}\right)$$

$$\psi = \frac{1}{2}\left(\frac{\gamma-\beta}{C+\alpha} + \sqrt{\frac{(\gamma-\beta)(B(\gamma-\beta) - C(3\beta+\gamma) - \alpha(\beta+3\gamma))}{(B-C+\alpha)(C+\alpha)^2}}\right)$$

which is a locally asymptotically stable if and only if

$$B\beta + C(3\beta+\gamma) + \alpha(\beta+3\gamma) < B\gamma.$$

Proof: We have seen before that the prime period-two solution $\{\varphi,\psi\}$ exists if and only if

$$B\beta + C(3\beta+\gamma) + \alpha(\beta+3\gamma) < B\gamma.$$

We will now show that this solution is always locally asymptotically stable when there exists. We have that

$$\mathbf{det} J_{T^2}(\varphi,\psi) = \frac{(C\beta+\alpha\gamma)(C^2\beta - BC\gamma - \alpha(B+\alpha)\gamma)}{(\beta-\gamma)(B^2(C+\alpha)\gamma - BC(C+\alpha)(\beta+\gamma) + C^2(C\beta+\alpha\gamma))}$$

and

$$\mathbf{tr} J_{T^2}(\varphi,\psi) = \frac{(-C^3 - C^2\alpha + 3C\alpha^2 + \alpha^3)\beta^2 + B^2(C+\alpha)\beta(\beta-\gamma) + (C-\alpha)^2(C+2\alpha)\beta\gamma}{(\beta-\gamma)(B^2(C+\alpha)\gamma - BC(C+\alpha)(\beta+\gamma) + C^2(C\beta+\alpha\gamma))} + \frac{(C^3+2C^2\alpha-C\alpha^2-4\alpha^3)\gamma^2+B(C+\alpha)((C+2\alpha)\beta^2-C\beta\gamma-(C+3\alpha)\gamma^2)}{(\beta-\gamma)(B^2(C+\alpha)\gamma-BC(C+\alpha)(\beta+\gamma)+C^2(C\beta+\alpha\gamma))},$$

and based on this we conclude that it is

$$|\mathbf{tr} J_{T^2}(\varphi,\psi)| < 1 + |\mathbf{det} J_{T^2}(\varphi,\psi)| < 2$$

if and only if

$$B\beta + C(3\beta+\gamma) + \alpha(\beta+3\gamma) < B\gamma.$$

3. Conclusion

The Wolfram Mathematica package was used to prove the existence and stability of the prime period-two solution, without which it would be quite difficult to report the appropriate conditions. For that reason, the existence of a period-two solution is not stated here in the case when all parameters are strictly positive. The obtained results represent an important contribution to the examination of the global dynamics of Eq. (1).

Eq. (1) in the case when $A = \alpha = 0$ was examined in detail in [7]. From the stability analysis of prime period-two solution of Eq. (1) as well as the existence and stability of a unique positive equilibrium, we see that only period-doubling (flip) bifurcation is possible. In one of the following papers it will be necessary to do a bifurcation analysis of Eq. (1). Something similar was done in [8] and [9]. In the case when the mapping T is competitive, it is possible to obtain quite precise results on the global dynamics of Eq. (1), which can be done in one of the following papers. See similar results in [10] and [11].

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