State Approach - Index - based Measurement

Josef Myslin, Jiri Kaiser

Faculty of Civil Engineering, Czech Technical University in Prague, Thakurova 7/2077, 166 29 Prague 6, Czech Republic

Abstract **– This work is focused on the state view in modeling processes or other systems and develops the basic theses postulated by the authors in previous publications. It focuses on understanding the need for measurement within state spaces and the individual basic methods of this measurement. The main goal of this work is to define an index-based approach to measurement, and thus to define states within the state approach. This work brings new view on measurement using indexes and their usage in definition of state among the state space.**

Keywords **– Index-based measurement, state approach.**

1. Introduction

The state approach to system modeling is one of the newer methods of approach to this engineering activity. It can cover gaps that appear in traditional approaches to modeling processes and systems. Theoretically, it is based on the theory of automata [1], but further develops the issue for the ability to model technical and other systems.

The practice of process and systems modeling shows that traditional approaches to modeling based on behavioural, structural or functional concepts, however useful, show philosophical shortcomings in modeling certain types of processes or systems [2]. These gaps have been appearing for a long time [3] and their solutions are usually modifications of existing methodologies [4]. The state approach in the form described in this article was also one such modification, namely the modification of behaviourally oriented process modeling methodologies [5].

The authors' own research in this area continued with efforts to constitute the foundations of state theory for the modeling of processes and systems [6]. Basic concepts and principles were introduced, which, however, were not supported by appropriate means for measurement and evaluation within state spaces and the systems modelled by them. This article therefore opens the necessary topic and approaches an answer to the question of how to define and how to determine the mutual distance of individual elements of the state space. By distance we mean the degree of practical difference between these elements.

2. Measurement in State Space, Definition of State Distance

▪ Motivation

In previous article [6], we formally defined the basic concepts, including the concept of state. Formally, therefore, we can define it very well. However, we have to ask ourselves how we will define the state in real practice, in a specific state space. We intuitively suspect that by state we will understand such a subset of state space in which the individual elements have certain common properties. In some cases, the set is defined explicitly, for example, based on a norm, based on law, and so on. In such cases, our situation is easy, the state is defined directly. In other cases, however, we may ask how we will define individual states. To group individual points of state space into subsets that will represent individual states, we have to be able to assess the similarity and distance of individual points of state space.

Another potential problem is the determination of the similarity and distance of individual points of

DOI: 10.18421/TEM112-03
https://doi.org/10.18421/TEM112-03

Corresponding author: Josef Myslin,
Faculty of Civil Engineering, Czech Technical University in Prague, Thakurova 7/2077, 166 29 Prague 6, Czech Republic.
Email: myslin@volny.cz

Received: 09 February 2022.
Revised: 17 March 2022.
Accepted: 23 March 2022.
Published: 27 May 2022.

state space, especially regarding the transition between individual points of state space, regarding the comparison of individual real situations or regarding the need for statistical evaluation (e.g., cluster analysis methods [7]).

▪ Intuitive concept of distance

It is therefore necessary to introduce a formal definition of how we will understand the distance (difference) between two points of state space. Obviously, there is no one way and different methods of understanding distance (difference) will be suitable for different purposes. Given how the state space was defined - i.e., as an n-dimensional space of possible values of definition variables - we can intuitively try to introduce the distance between two points of this space as a common Euclidean distance [9] in the Cartesian coordinate system - see Figure 1.

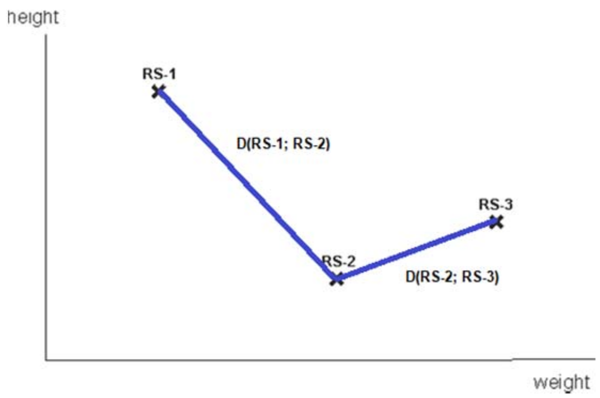


Figure 1. Intuitive Euclidean distance

So, distance between two real situations (between two points of state space) can be defined in this way (assuming the existence of state space S with definition variables x1, x2 and real situations RSA, RSB from S).

$$D(RS_A, RS_B) = \sqrt{(x_{1A} - x_{1B})^2 + (x_{2A} - x_{2B})^2}$$

for two-dimensional state space. For the general n-dimensional state space, the general n-dimensional relation for the Euclidean distance will be used:

$$D(RS_A, RS_B) = \sqrt{(x_{1A} - x_{1B})^2 + (x_{2A} - x_{2B})^2 + \cdots + (x_{NA} - x_{NB})^2}$$

This way of introducing the distance between two points of state space is simple and easy to understand, but unfortunately it presents several fundamental problems that can appear when describing a real system. One of them is that the distance between two points of state space in the form of a simple Euclidean distance can only be introduced on an orthonormal state space. For simplicity, we understand the state space strictly as an n-dimensional Cartesian space - this guarantees us the orthogonality of the state space as one of the conditions of orthonormality [10]. The second condition is the unit size of the base vectors, which in our case can be simply understood as a condition of identical units on all axes of the state space (of size 1). If the definition properties of the state space are a description of the same quantity (for example, length, mass, etc.), then the condition sufficient for the orthonormality of the state space will be the fact that we express the individual definition properties in the same units. The problem occurs in the case of unequal quantities, because then there may be (and is) a situation where the individual definition properties are not comparable. We can see Figure 2 - the Euclidean distance between RS1 and RS2 and between RS2 and RS3 in this case is the same. However, the question is whether in the case of different quantities of definition properties (weight and height) the Euclidean distance can be used in this way at all - i.e., whether the difference between 1 cm and 1 kg is the same. At first glance, it is evident that such an interpretation is quite problematic, even though it is not possible to unambiguously determine the unit in which we measure distance.

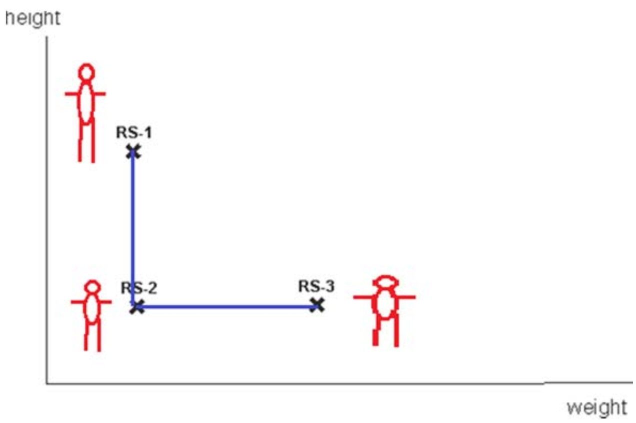


Figure 2. Problems with distance

3. Transformation of Definition Properties

In some cases, the incompatibility of defining quantities can be removed and the Euclidean distance (or another method of distance definition) of individual points of the state space can be introduced in the state space, as described in the previous paragraphs. A necessary condition is that we can philosophically define this notion of distance and its meaning in each situation - but this is an aspect that has to be decided by those who assess the system.

If the only problem of the considered system is the fact that the individual definition properties of the state space are different quantities, but we can imagine their transformation so that we can then claim that their unit changes are comparable, we can introduce the following transformation:

$$\overline{x_n} = f(x_n)$$

where $\overline{x_n}$ is nth transformed definition property. The condition for introducing such a transformation is the existence of a function f on the whole domain of the nth definition property. However, it is important to perceive that the transformation function also transforms the units in which the quantity is expressed, which means that the transformed defining property can no longer be understood in the original units (kilograms, meters, etc.), but as a dimensionless quantity. It is then necessary to understand all dimensional properties as dimensionless - where we do not want to perform a transformation, we will consider the transformation by a unit function. By converting all quantities to dimensionless, we technically achieve that all definition properties are described in the same way.

4. Index on State Space

In some cases, however, we find that the very principle of Euclidean distance for a given state space does not make sense, and therefore does not make sense to consider transformations of definition properties at all. This is, for example, the case of two-dimensional state space mentioned by us, where the definition properties are height and weight. In such a case, we have to look for other means that will allow us to overcome the incompatibility of definition properties. One possibility is an index, which we will understand as follows:

$$I_S = f(x_1, x_2, \cdots, x_N)$$

Where S is state space and x_k are definition properties. The condition for introducing this index is the existence of a function f on the whole state space. The index therefore replaces the definition properties with a single number. This number is dimensionless and is obtained based on a function of n variables, which are the defining properties of a given state space. It is obvious that the quality of the representation of the properties of a state space element by this index depends on the appropriate construction of the indexing function. It has to describe in an appropriate way the interrelationships and connections between the definition variables. As an example, we can mention a common index, which is used in the state space described by us (weight, height) - it is the so-called body mass index (BMI), which is defined as

$$BMI = \frac{weight}{height^2}$$

This index makes it possible to compare individual elements of the state space (but also individual persons and the real situations defined by them) – see Figure 3.

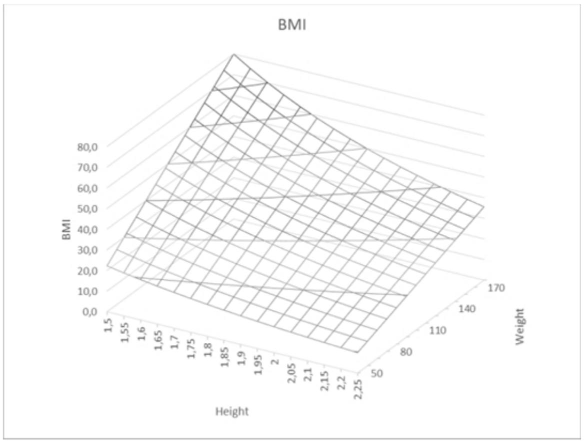


Figure 3. Body mass index

It is also important to understand that any number of indexes can be defined on a given state space, with each index indicating a state space element in a different way. The index is therefore strictly dependent on the context, so even an index that is very suitable for one problem may prove unusable for another problem.

5. Use of Index-Based Distance in State space

If we have a defined distance in the state space (i.e. if we can determine how the individual points of the state space differ in a given sense), either by standard Euclidean distance, Euclidean distance in dimensionless transformed state space, index or other methods (methods described in this article is far from an exhaustive list), we can tell many facts about state space and real situations.

▪ General

For example, it is possible to make a real comparison of two or more real situations, or to assess the difficulty of the transition between them. The possibility of comparison is a necessary condition for the possibility of statistical evaluation, or for performing various analyses, such as cluster analysis [8].

Finally, it should be noted that the term distance can be used for the definition of individual states. As already mentioned, the state can be defined explicitly i.e., on the basis of some external stimulus (legislation, decision), but we can also define it precisely on the basis of distance or otherwise conceived evaluation. By state we mean a set of points of state space that satisfies the given criteria (distance from a point, index value, etc.). Again, however, it has to be stated that even the best set measurement methodology will not work without knowledge of the system we are trying to model. The state, if it is to have meaning and a real image in the modeled reality, has to be based on knowledge of the system.

▪ Index-based state distance

So far, we have understood the concept of distance in terms of length, either directly i.e., as the actual length of the "route" between two points of the state space, or indirectly, as the length between two points in the state space graph. Indices allow us to introduce another way of looking at distance, namely distance in terms of similarity. We will not determine the distance in the physical sense, in terms of length, but in terms of differences in index values. In other words, within state space, two points may be far apart, but their distance (in the sense of similarity, in the sense of close index values) may be very small or even zero.

For example, on the already mentioned map, two places can have the same temperature, and thus have zero distance within this index (temperature index). This is said as follows - we have introduced a temperature index, because temperature is what we want to characterize a given place (a given element of state space). Therefore, if two places have the same temperature, they are equivalent for us in this view. Similarly, we are introducing BMI (Body Mass Index) to compare individuals with different heights and weights - in this case we are interested in body mass, and here it is possible that in this parameter the dwarf and the giant will have the same value. Thus, their distance in terms of body mass will be zero, because body mass will be the same. If we were to prove these zero distances in practice, try to imagine that different people come to the doctor's office - the doctor finds out their BMI and makes a recommendation accordingly. The mentioned giant and the dwarf (they have the same BMI) will receive the same recommendation - to stay on the given weight, increase body weight, reduce body weight.

Two remarks have to be made on the concept of distance in the sense of the similarity of the values of the index. The first concerns a certain discomfort in use - this discomfort results from the fact that we usually perceive the concept of distance in the already mentioned sense of length, which is understandable and intuitive. It should be noted that in most state spaces, there are no definition properties having an aspect of length at all, and that length appears only virtually in a possible state space graph. For this reason, it will not be a problem to get used to the perception of distance in terms of the similarity of the index value. The second remark then says that the distance is strictly tied to a given index - when changing the index, the distances (similarities) will be completely different and there is no doubt that many pairs close within one index will be very far within the other index. For example, two places may have the same temperature but completely different pressures.

So, we can define the simple distance of two real situations in state space based on the index as follows

$$\mathfrak{D}_{\sigma}(RS_1, RS_2) = |\sigma(RS_1) - \sigma(RS_2)|$$

where σ is selected index. We use absolute value to ensure the commutativeness of the operation. The distance is expressed in units of the index. We speak of a simple distance because it is a simple difference between the values of the index, considering neither the characteristics of the index nor the characteristics of the course of the index, which may not be linear between two points of state space. We can cover some characteristics of the index by subtracting the transformed values instead of simple index ranks, for example within a logarithmic scale. Then the formula would look like this.

$$\mathfrak{D}_{\sigma}(RS_1, RS_2) = |\log(\sigma(RS_1)) - \log(\sigma(RS_2))|$$

or general

$$\mathfrak{D}_{\sigma}(RS_1, RS_2) = |\mathrm{f}(\sigma(RS_1)) - \mathrm{f}(\sigma(RS_2))|$$

where f is transformation function. As an example, we present the following map on Figure 4, which describes the ordinary world as a state space, the index being the altitude, which assigns each point of the state space an altitude from sea level. In the excerpt from the map, we have two real situations (i.e., two places where we can be), these are named High-mountain and Low-mountain.

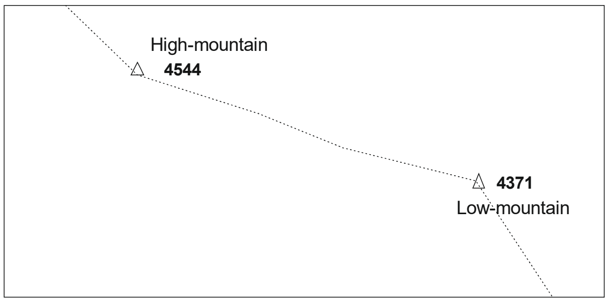


Figure 4. Part of a fictive map

$$\sigma(RS_1 - Highmountain) = 4544\ m$$
$$\sigma(RS_2 - Lowmountain) = 4371\ m$$

Then we can express the distance of two real situations (two vertices) in the concept of altitude index as follows

$$\mathfrak{D}_{\sigma}(RS_1, RS_2) = |\sigma(RS_1) - \sigma(RS_2)| = |4544 - 4371| = 173\ m$$

However, it should be noted that the actual "height profile" of the area is approximately as shown in Figure 5. A simple distance solves only the absolute difference - in many cases, this simple distance is sufficient, if not, it is necessary to choose more complex indicators that also consider the course of the path between two real situations within the state space.

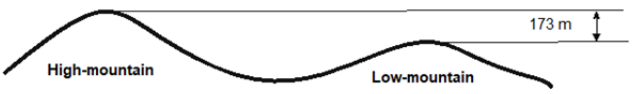


Figure 5. Fictive height profile

For more detailed information not only about the simple distance within a given index, but also about the structure of the state space, we can successfully use isolines.

▪ Index-based isolines

We can understand the index as a very reasonable and relatively easy-to-use tool. The index assigns one value to each state space point, which greatly simplifies the understanding of the situation within the state space. If we can find a suitable index or a suitable index function, we can easily define the distance of individual elements of the state space.

As useful tool we can consider finding state space elements for which the value of a given index is the same - although these elements may be remote within the state space, they show some identity in terms of a given index. Here it is necessary to reiterate the need to introduce a meaningful and adequate index - only in this case does it make sense to compare the index and perceive it as a relevant indicator. As an example, we can mention a meteorological map - its basis is a standard map of a given area, in which the value of one of the meteorological quantities is marked - this quantity is an index of its kind, because for each point of state space (given by map coordinates) one number is assigned (meteorological quantity - temperature, pressure, ...).

The same value of the index (meteorological quantities) says that the two elements of the state space (world, maps) show similar natural conditions in the given sense - if, for example, we know that two places have the same temperature, we can deduce for example the simple fact that in these places we can choose similar clothes. Of course, experts in the field of meteorology use more sophisticated considerations about given natural events.

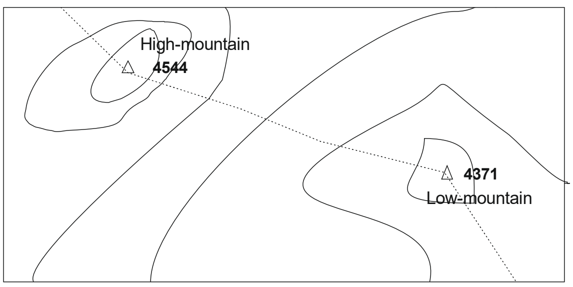


Figure 6. Fictive geographic map with contour lines

For these considerations, it is often useful to find not two places with the same values, but whole sets of places that have the same index value. If we can connect these places into a curve, then we call this curve isoline - some isolines have their own specific names, for example isobars as curves connecting places with the same pressure, isotherms as curves connecting places with the same temperature, or contour lines as curves connecting places with the same altitude. Fictive map is in Figure 6.

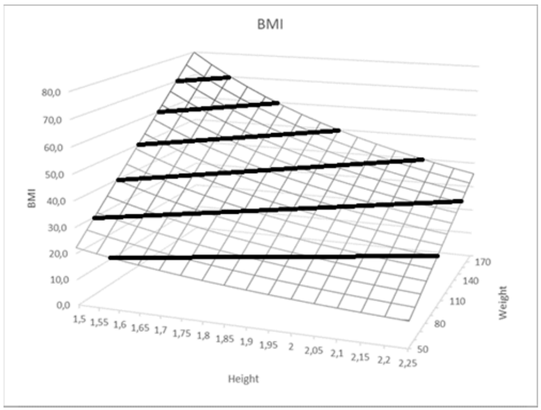


Figure 7. BMI chart with highlighted isolines – 3D view

In Figure 7 and Figure 8 BMI is shown as an example of real and practical index and its isolines. Of course, this approach can be used for any index - only for indices that are not themselves directly physical quantities (temperature, pressure, height, current, etc.) it is necessary that the constructed index makes sense in the real world. However, this is a characteristic of the index itself - we have already reported on this necessary condition in the section focused directly on the definition and construction of index.

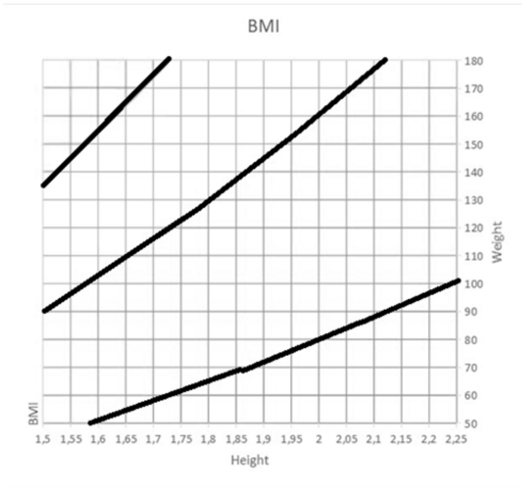


Figure 8. BMI chart with highlighted isolines – top view

The use of isolines corresponds to the use of contour lines on a classical geographical map or the use of isobars on a meteorological map. Thanks to these lines, we have an easy visual overview of the structure of the state space (within the given selected index). We can estimate the gradient i.e., the degree of "rise" of the index - and thus, in a given sense, the difficulty of movement between individual points of the state space. An appropriately chosen index will then allow us to directly define states i.e., subsets of state space.

▪ Index-based state definition

Recall that we define a state this way:

$$ST \subset S$$

ST is any subset of basic state space S. The method of defining a given subset determining the state is arbitrary. It is therefore possible to define the state also based on defining the values of the selected index. The definition is

$$ST \subset S;\ \forall x \in ST: x \in S\ \wedge\ \sigma(x) \in \mathcal{S}$$

where S is the definition set of the index. If we again use the already mentioned BMI index and we want to define the state as a space in which the index values are between 19 and 25, then the definition set will be as follows:

$$\mathcal{S} = \langle 19; 25 \rangle$$

the corresponding state will then look as shown in Figure 9.

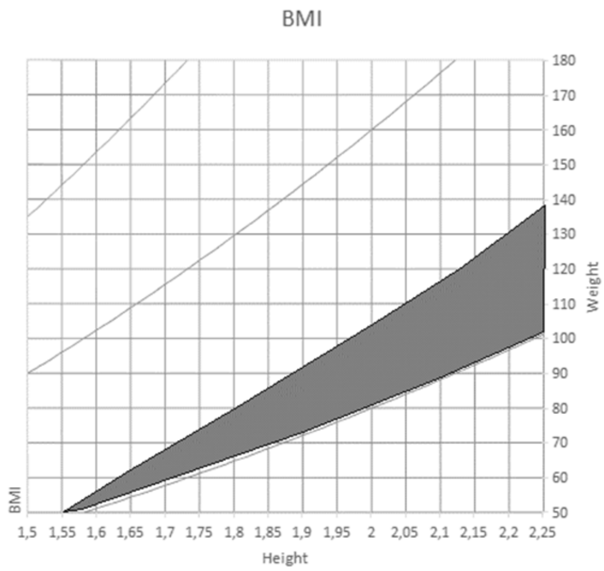
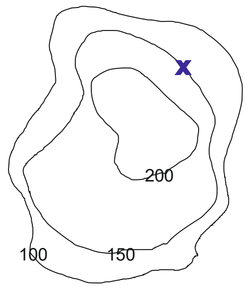
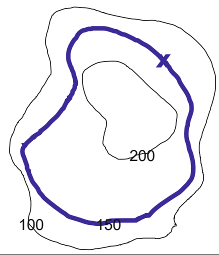
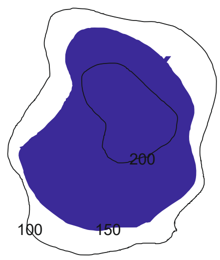
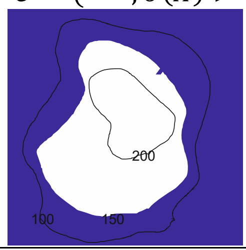
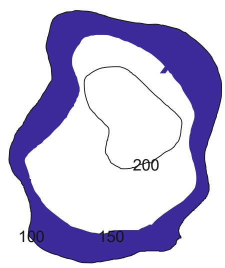
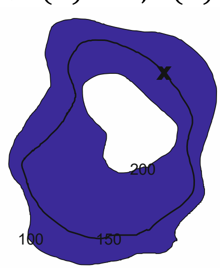


Figure 9. Index based defined state

Such a state can be (and in practice we do) called the state "normal weight in relation to height".

The definition set of the index for the state definition can be completely arbitrary, and Table 1 shows the basic variants. Of course, other variants may appear depending on the needs of the practice - however, it is necessary to realize that for practical use it is necessary that both the index itself and the defined state have a practical content and meaning.

Table 1. Possible forms of state definitions

Type	Picture
Basic map of state space with isolines	
Set of equal values **ISOLINE**	$\mathcal{S} = \{\sigma(X)\}$ 
Set of higher values	$\mathcal{S} =< \sigma(X); \infty)$ 
Set of lower values	$\mathcal{S} = (-\infty; \sigma(X) >$ 
Interval	$\mathcal{S} =< A; B >$ 
Ring	$\mathcal{S} =< \sigma(X) - D; \sigma(X) + D >$ 

The states defined in this way can then be used for other activities within state-defined approaches, for example for modeling transitions between states or for further measurements using, for example, more advanced mathematical methods (for example, differential or integral calculus or statistics).

6. Discussion

It follows from the text (and especially from practice) that the two basic problems of the state approach are the definition of states and measurements within the state space. Both problems often result from the incommensurability of defining quantities, where it is difficult to work with often diametrically different types of quantities. Of course, these reasons are not the only ones, but they are undoubtedly crucial.

The ability to compare different elements of the state space i.e., the real situation within the state space is crucial. In the previous text, we showed as an example the state space defined by the weight and height of a person. The question then was whether the individual could be compared. In the basic concept of state space, such a comparison is relatively difficult - however, the situation changes fundamentally when using the index. The index reduces the values of individual definition variables into one numerical value, which can then be compared with other values of the same index. Each element of the state space is thus further described not by n-tuple of values (as already mentioned, often difficult to compare - height / weight), but by a single number, an index.

▪ Index quality

However, another question arises - is there a single index above a given state space or is there a larger number of indexes? If we focus on the definition of an index, which speaks of the fact that the index is a function of the definition values of the state space, then it follows from the logic that there are an infinite number of possible indexes over a given state. Of course, deciding how to construct an index function is not easy. We start from two basic aspects.

The first is the question to which index should provide an answer. The value of the index tells us what the given element of the state space is - but the context is very important. If we used the BMI (body mass index) index as an index in the height / weight state space, then we asked the question what the ratio of weight and height is, or how does a person's height correspond to that person's weight. From this point of view, we can state that two people can be comparable, even though their heights and weights are diametrically different (for example, both people may be slim or obese). In another context, however, these people are not comparable, and we will use a different index for this different context. Within this context, we have to identify the basic approach to constructing an index function definition. Thus, for example, the BMI index is constructed as a ratio of mass and height (now we do not solve multiplicative or power factors), other indices may be constructed as summation, multiplicative, logarithmic, etc.

The second factor is the experience and knowledge of state space as a description of part of reality. There is no doubt that the first attempt to construct an index may not lead to the definition of such an index, which is well suited as a tool for comparing state space elements. If we construct an index that describes whether a person is slim or obese within the state space height / weight, it is evident that we have to compare the values of these defining properties, i.e., that it will be a share index. But it is not clear whether we should use, for example, multiples or powers of the values of individual fields. Here, the experience and knowledge of state space best serve as a description of a part of reality.

▪ Index construction

It follows from the above that it is basically impossible to construct a quality index without a well-answered question of what the index should tell us and in what context we want to use it. And, without knowledge of state space, without knowing how the defining variables relate to each other.

If we were to think about the mentioned BMI index, then we have already agreed that it will be a ratio index. However, if we only divided the weight by height, we would not get completely correct values. The reason is that height and weight cannot grow in the same way - if one person is half as tall, he will certainly be (at the same body proportions) more than half as heavy. Height increases linearly, but the corresponding mass increases approximately to the square of the square - see Figure 10. This relationship between weight and height is the reason why in the formula for calculating the index, the height in the denominator is raised to the power of the second.

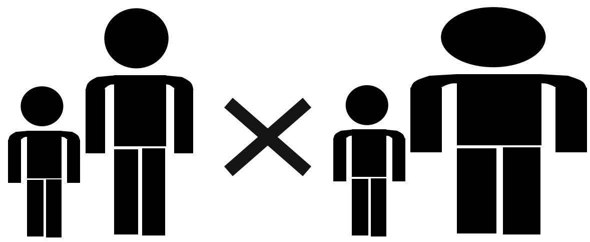


Figure 10. People with the same proportion versus people with different proportion

After constructing the index, it is also appropriate to verify it in practice - ideally on the elements of the state space, which we intuitively understand as comparable (in the given context). Despite all such control, it may happen that the index in practice will not work exactly as expected - then it is necessary to make larger or smaller corrections or choose a completely different index.

▪ Index usability

Finally, the question arises whether the index is a better indicator of similarity and distance within the state space than a direct comparison of the values of the defining properties. In general, if the defining properties are comparable (ideally the same quantity), then a direct comparison can be used relatively easily. Usually, however, this is not the case, and in such a case, a suitably chosen index is undoubtedly a benefit, because we compare a single number. The connection "appropriately chosen index" is important. On the contrary, the index chosen inappropriately distorts the comparison and the results are then bad.

A well-chosen index undoubtedly contributes to a better possibility of comparing real situations and, in general, to an orientation within the state space. There is no doubt that obtaining a quality index requires experience, knowledge, and in many cases also repeated testing - it follows that such activity usually only makes sense for those state spaces that we will address in the long run, repeatedly, or that are important to us. In such cases, however, it certainly makes sense to consider, construct and calculate indices. In the optimal case, the entire index set can be defined on a frequently used state space, where the individual indexes can answer various questions concerning the given state space. It is then even possible to construct indices over indices or to compare the values of individual indices for individual elements of the state space.

7. Conclusion

Measurements within the state space prove to be necessary not only for fundamental understanding, but also for many applications such as the very definition of states, but also for statistical evaluation of real situations in each system. This paper presents an introductory insight into the issue with the identification of several basic solutions. These solutions should be further developed for specific applications. For example, we can talk about the need to introduce a measure of effort into a state diagram, and then define the distance on curved state spaces. We also find it important to continue the development of statistical data processing.

We have shown that it is possible to successfully use an index i.e., from a mathematical point of view, a function of several variables (state space definition variables), which assigns one characterizing number to each element of the state space. This index can then be used both to measure the distance (in the sense of similarity) of individual points of the state space, as well as to directly define states as subsets of the state space having characteristic properties in the sense of the given index. Of course, everything provided a correctly and correctly set index. The indices also provide the possibility of further evaluation of the properties of the state space, which will be the subject of further work.

Acknowledgements

This work was supported by the Grant Agency of the Czech Technical University in Prague, grant No. SGS22/139/OHK1/3T/11.

References

[1]. Alur, R., & Dill, D. L. (1994). A theory of timed automata. *Theoretical computer science*, *126*(2), 183-235.

[2]. Wohed, P., van der Aalst, W. M., Dumas, M., ter Hofstede, A. H., & Russell, N. (2006, September). On the suitability of BPMN for business process modelling. In *International conference on business process management* (pp. 161-176). Springer, Berlin, Heidelberg.

[3]. Becker, J., Rosemann, M., & Uthmann, C. V. (2000). Guidelines of business process modeling. In *Business process management* (pp. 30-49). Springer, Berlin, Heidelberg.

[4]. Börger, E. (2012). Approaches to modeling business processes: a critical analysis of BPMN, workflow patterns and YAWL. *Software & Systems Modeling*, *11*(3), 305-318.

[5]. Myslín, J., & Kaiser, J. (2017). State based milestones in process modeling methods based on behavioral approach. *Education Excellence and Innovation Management through Vision 2020: From Regional Development Sustainability to Global Economic Growth*, *1*, 3598-3606.

[6]. Myslín, J., & Kaiser, J. (2018). Philosophical and technical aspects of term" state" in technical practice. *Journal of Systems Integration (1804-2724)*, *9*(2).

[7]. Edwards, A. W., & Cavalli-Sforza, L. L. (1965). A method for cluster analysis. *Biometrics*, 362-375.

[8]. Kettenring, J. R. (2006). The practice of cluster analysis. *Journal of classification*, *23*(1), 3-30. https://doi.org/10.1007/s00357-006-0002-6

[9]. Liberti, L., Lavor, C., Maculan, N., & Mucherino, A. (2014). Euclidean distance geometry and applications. *SIAM review*, *56*(1), 3-69.

[10]. Plonka, G. (1997). Necessary and sufficient conditions for orthonormality of scaling vectors. In *Multivariate approximation and splines* (pp. 205-218). Birkhäuser, Basel.