Coexistence between predator and prey in the modified Lotka - Volterra model

Vahidin Hadžiabdić [1], Midhat Mehuljić [1], Jasmin Bektešević [1], Naida Mujić [2]

[1] *Faculty of Mechanical Engineering, Department of Mathematics, University of Sarajevo, Sarajevo, Bosnia and Herzegovina*
[2] *Faculty of Science, University of Sarajevo, Sarajevo, Bosnia and Herzegovina*

Abstract **– Here we examine the behavior of a rational Lotka - Volterra model which is a modification of the ordinary polynomial case. We find nonnegative equilibrium points and define conditions in the parametric space for the stable positive equilibrium point. We also prove existence of the stable limit cycle in the case of the unstable positive equilibrium point.**

Keywords **– Equilibrium Point, Lotka - Volterra, Rational, Stable Limit Cycle, Stability.**

1. Introduction

Lotka - Volterra models have been observed in several papers. There are many variations of this model with a various numbers of predators and preys. Certain results in general case, Lotka-Voltera model with m predators and n preys by "Pure-Delay Type" Systems, are presented in [1]. In the case of a three-dimensional system it is possible to occur more than one limit cycle (see [2]). The basic Lotka - Volterra model with one predator and one prey is given by the system

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Corresponding author: Vahidin Hadžiabdić,
Faculty of Mechanical Engineering, Department of Mathematics, University of Sarajevo, Sarajevo, Bosnia and Herzegovina
Email: hadziabdic@mef.unsa.ba

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$$\begin{cases} \dot{x} = ax - bxy \\ \dot{y} = cxy - dy, \end{cases} \tag{1}$$

where x is a prey and y is a predator. One of the basic results for this model is the following theorem, (see [3]).

Theorem 1: *Every solution of the Lotka - Volterra system (1) is a closed orbit (except the equilibrium point and the coordinate axes).*

Theorem 1 implies existence of the periodic solution of the system (1) for all initial conditions, (see [4]). There is also a modified Lotka - Volterra model with a prey and two predators, where there is possible coexistence between them, under certain conditions for positive parameters.

$$\begin{cases} \dot{x} = ax - xy - xz \\ \dot{y} = -by + xy \\ \dot{z} = -cz + xz. \end{cases}$$

2. Rational Lotka - Volterra model

In this paper we consider the following rational Lotka - Volterra model:

$$\begin{cases} \dot{x} = x(1-x) - \frac{axy}{x+c} \\ \dot{y} = by(1 - \frac{y}{x}), \end{cases} \tag{2}$$

where x is a prey and y is a predator. In the system (2) the initial conditions are non-negative and the parameters a, b and c are positive real numbers.
The next theorem characterizes two equilibrium points.

Theorem 2: *System (2) has two non-negative equilibrium points, a saddle point $E_1(1,0)$ and a positive point*

$$E_2\left(\frac{1-a-c+\sqrt{4c+(a+c-1)^2}}{2}, \frac{1-a-c+\sqrt{4c+(a+c-1)^2}}{2}\right),$$

where a,b and c are positive real parameters. Also, x-axis is a stable manifold of point $E_1(1,0)$.

Proof: Equilibrium points are the solutions of the following system

$$\begin{cases} x(1-x) - \dfrac{axy}{x+c} = 0 \\ by\left(1 - \dfrac{y}{x}\right) = 0. \end{cases}$$

The second equation implies $y_1 = 0$ and $y_2 = x$ what along with the first equation gives $x_1 = 1$ and

$$x_{2,3} = \frac{1-a-c \pm \sqrt{4c+(a+c-1)^2}}{2}.$$

Since

$$x_3 = \frac{1-a-c-\sqrt{4c+(a+c-1)^2}}{2} < 0,$$

there are only two non-negative equilibria.

The system (2) can be written in the form

$$\dot{x} = f(x),$$

where

$$f(x) = \begin{pmatrix} x(1-x) - \dfrac{axy}{x+c} \\ by\left(1-\dfrac{y}{x}\right) \end{pmatrix}.$$

Jacobian matrix of $f(x)$

$$D_f = \begin{pmatrix} 1 - \dfrac{ay}{c+x} + x\left(\dfrac{ay}{(c+x)^2} - 2\right) & \dfrac{-ax}{c+x} \\ \dfrac{by^2}{x^2} & b - \dfrac{2by}{x} \end{pmatrix}$$

evaluated at the point $E_1(1,0)$ is given by

$$D_f(E_1) = \begin{pmatrix} -1 & \dfrac{-a}{c+1} \\ 0 & b \end{pmatrix}.$$

The corresponding eigenvalues of $D_f(E_1)$, $\lambda_1 = -1$ and $\lambda_2 = b$, have an opposite sign so the point $E_1(1,0)$ is a saddle point for all parameter values (see [5]).

Obviously the both components of the point $E_2\left(\frac{1-a-c+\sqrt{4c+(a+c-1)^2}}{2}, \frac{1-a-c+\sqrt{4c+(a+c-1)^2}}{2}\right)$ are positive.

For $y = 0$ system (2) is reduced to the logistic equation

$$\dot{x} = x(1-x)$$

whose solution is

$$x = \frac{1}{1+Ce^{-t}}.$$

Hence $x \to 1$ when $t \to \infty$ and the x-axis is a stable manifold of the equilibrium point $E_1(1,0)$. For more details of the logistic equation see [6,7].

We use Tr - Det terminology to prove stability of the equilibrium point E_2 under certain conditions, (see [3,8]). The eigenvalues of the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

are solutions of the equation:

$$\lambda^2 - (a_{11}+a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0,$$

or equivalently

$$\lambda^2 - (tr(A))\lambda + \det(A) = 0.$$

So,

$$\lambda_{1,2} = \frac{tr(A) \pm \sqrt{(tr(A))^2 - 4\det(A)}}{2}.$$

The following Lemma shows $\det(D_f(E_2))$ is strictly positive.

Lemma 1: *If* $a,b,c > 0$ *then*

$$det\left(D_f(E_2)\right) > 0.$$

Proof: The Jacobian matrix D_f evaluated at the point E_2 is

$$D(E_2) = \begin{pmatrix} d_{11} & \dfrac{-1-a-c+\sqrt{4c+(a+c-1)^2}}{2} \\ b & -b \end{pmatrix},$$

where

$$d_{11} = \frac{2a-1+3c-2\sqrt{a^2+2a(-1+c)+(1+c)^2)}}{2} - \frac{(1+c)(-1-c+\sqrt{a^2+2a(-1+c)+(1+c)^2})}{2a}.$$

Since $a,b,c > 0$ and

$$\det\left(D_f(E_2)\right) = \frac{b(-a+2-2c+\sqrt{a^2+2a(-1+c)+(1+c)^2})}{2}$$
$$+\frac{b(1+c)(-1-c+\sqrt{a^2+2a(-1+c)+(1+c)^2})}{2a}.$$

we only need to prove

$$(a+1+c)\sqrt{a^2+2a(-1+c)+(1+c)^2}$$
$$>(1+c)^2+a^2-2a+2ac,$$

or

$$(a+1+c)\sqrt{(a+c+1)^2-4a} > (a+c+1)^2-4a.$$

Expression $(a+c+1)^2-4a$ is strictly positive. Indeed,

$$(a+c+1)^2-4a = (\mathrm{a}+\mathrm{c}-1)^2+4\mathrm{c}\,,$$

so

$$(a+1+c) > \sqrt{(a+c+1)^2-4a}.$$

Hereof

$$\det\left(D_f(E_2)\right) > 0.$$

Lemma 2: *If* $a,b,c>0$ *and*

$$(2a+c+1)\sqrt{(a+c-1)^2+4c}+a+2ab$$
$$>2a^2+(c+1)^2+3ac$$

then

$$tr\left(D_f(E_2)\right) < 0.$$

Proof:

$$\mathrm{tr}\left(D_f(E_2)\right)$$
$$=\frac{2a^2-(1+c)(-1-c+\sqrt{a^2+2a(-1+c)+(1+c)^2}}{2a}$$
$$-\frac{a(1+2b-3c+2\sqrt{a^2+2a(-1+c)+(1+c)^2}}{2a}$$

and $\mathrm{tr}\left(D_f(E_2)\right) < 0$ iff

$$\frac{2a^2-(1+c)(-1-c+\sqrt{a^2+2a(-1+c)+(1+c)^2}}{2a}$$
$$<\frac{a(1+2b-3c+2\sqrt{a^2+2a(-1+c)+(1+c)^2}}{2a},$$

what is equivalent to

$$(2a+c+1)\sqrt{(a+c-1)^2+4c}+a+2ab$$
$$>2a^2+(c+1)^2+3ac.$$

This proves our *Lemma*.

Theorem 3: *If* $a,b,c>0$ *and*

$$(2a+c+1)\sqrt{(a+c-1)^2+4c}+a+2ab$$
$$>2a^2+(c+1)^2+3ac$$

then the equilibrium point E_2 *is stable.*

Proof: We need to prove both eigenvalues

$$\lambda_1 = \frac{tr\left(D_f(E_2)\right)-\sqrt{\left(tr\left(D_f(E_2)\right)\right)^2-4\,det\left(D_f(E_2)\right)}}{2},$$

$$\lambda_2 = \frac{tr\left(D_f(E_2)\right)+\sqrt{\left(tr\left(D_f(E_2)\right)\right)^2-4\,det\left(D_f(E_2)\right)}}{2},$$

of the matrix $D_f(E_2)$ are negative or have negative real parts.

According to *Lemmas 1* and *2* if

$$\left(tr\left(D_f(E_2)\right)\right)^2-4\,det\left(D_f(E_2)\right)>0,$$

then

$$tr\left(D_f(E_2)\right)-\sqrt{\left(tr\left(D_f(E_2)\right)\right)^2-4\,det\left(D_f(E_2)\right)}<0.$$

Similarly

$$tr\left(D_f(E_2)\right)+\sqrt{\left(tr\left(D_f(E_2)\right)\right)^2-4\,det\left(D_f(E_2)\right)}<0$$

if

$$\left(tr\left(D_f(E_2)\right)\right)^2-4\,det\left(D_f(E_2)\right)<0.$$

This implies both eigenvalues are complex with negative real part and the equilibrium point E_2 is stable under the above stated conditions.

Theorem 4: *If* $a,b,c>0$ *and*

$$(2a+c+1)\sqrt{(a+c-1)^2+4c}+a+2ab$$
$$=2a^2+(c+1)^2+3ac$$

then the equilibrium point E_2 *is a center.*

Proof: Similarly as in *Lemma 2* we can prove

$$tr\left(D_f(E_2)\right) = 0 \qquad (3)$$

for

$$(2a+c+1)\sqrt{(a+c-1)^2+4c}+a+2ab$$
$$=2a^2+(c+1)^2+3ac.$$

Now we obtain pure imaginary eigenvalues

$$\lambda_{1,2} = \pm i\sqrt{\det\left(D_f(E_2)\right)}$$

and the equilibrium point E_2 is a center (see[5]).

Theorem 5: *If* $a, b, c > 0$ *and*

$$(2a+c+1)\sqrt{(a+c-1)^2+4c}+a+2ab$$
$$<2a^2+(c+1)^2+3ac$$

then the equilibrium point E_2 *is unstable. Furthermore, there is a stable limit cycle.*

Proof: The proof of the inequality

$$2a^2b+(1+b)(c+1)^2+2ac+3abc < a(b^2+b+2)$$

for $a, b, c > 0$ is similar to the proof of *Theorems 3* and *4*. It implies

$$tr\left(D_f(E_2)\right) > 0$$

what leads to the conclusion that the equilibrium point E_2 is unstable.
The limitation of the prey x follows from the first equation of the system (2), since the function $x - x^2$ is decreasing. Same, the function $by - \frac{by^2}{x}$ of the second equation in (2) is decreasing by y what obtains limitation of the predator y. Thus, both variables x and y are bounded in the first quadrant. Since both equilibrium points E_1 and E_2 are unstable there is a stable limit cycle.

In the following we present a graphical illustration of Theorems 3 and 5.

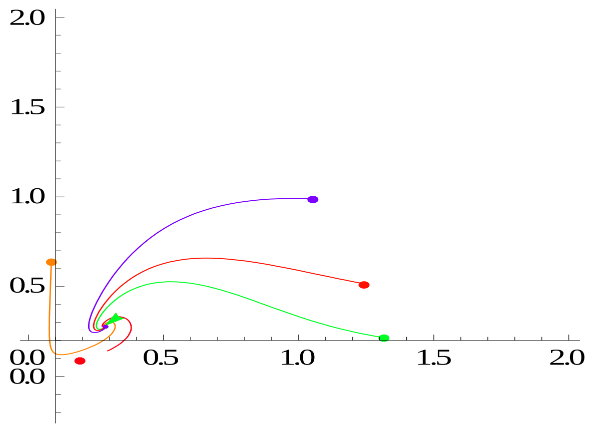

Figure 1. The case when the equilibrium point E_2 is stable for the parameter values a=2, b=1 and c=0.5.

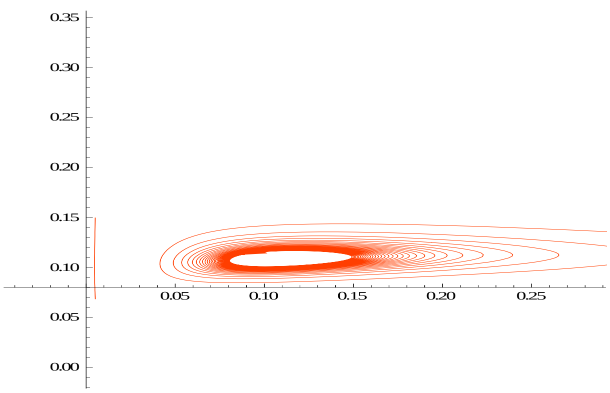

Figure 2. The case when the equilibrium point E_2 is ustable for the parameter values *a=5*, *b=0.05* and *c=0.5*.

3. Conclusion

Depending on the values of the parameters a, b and c there are 5 dynamic scenarios for the system (2) which includes the cases of the stable equilibrium point and the unstable point with a stable limit cycle. Depending on whether it is $\left(tr\left(D_f(E_2)\right)\right)^2 - 4\,det\left(D_f(E_2)\right) < 0$ or $\left(tr\left(D_f(E_2)\right)\right)^2 - 4\,det\left(D_f(E_2)\right) \geq 0$, there are 5 possible scenarios for the point E_2:

1. If $tr\left(D_f(E_2)\right) < 0$ and $\left(tr\left(D_f(E_2)\right)\right)^2 - 4\,det\left(D_f(E_2)\right) > 0$ the equilibrium point E_2 is a real sink.
2. If $tr\left(D_f(E_2)\right) < 0$ and $\left(tr\left(D_f(E_2)\right)\right)^2 - 4\,det\left(D_f(E_2)\right) < 0$ the equilibrium point E_2 is a spiral sink.
3. If $tr\left(D_f(E_2)\right) > 0$ and $\left(tr\left(D_f(E_2)\right)\right)^2 - 4\,det\left(D_f(E_2)\right) > 0$ the equilibrium point E_2 is a real source.

4. If $tr\left(D_f(E_2)\right) > 0$ and $\left(tr\left(D_f(E_2)\right)\right)^2 - 4\,det\left(D_f(E_2)\right) < 0$ the equilibrium point E_2 is a spiral source.
5. The case $tr\left(D_f(E_2)\right) = 0$ is unique and the equilibrium point E_2 is a center.

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