Identification of the Most Efficient Factors that Explain Confrontations Faced by Health Workers by Using a Two-Factor, Cross-Classification Linear Model with Applications

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***Abstract* – The violence that occurs in medical institutions include verbal threats, menacing behaviour, physical attacks, or sexual assaults, which are frequently initiated by patients, their relatives, or other individuals. These acts generate risks for health service workers. Workers in this field are attacked sixteen times more frequently than workers in other fields. In this study, a two-factor cross-classification model was applied, utilizing qualitative data, to identify factors that can provoke confrontations (violence). There are ten levels of factors used in the model. We identified the most common factor (given a certain probability range) as well as the best estimator and single-solution vector.**

***Keywords* – Linear model, variance analysis, two-factored cross-classification, factorial design, modelling.**

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1. Introduction

The variance analysis (VA) linear model, $Y = X\beta + \varepsilon$ which is used to analyse qualitative data, can also be written as a normal equation, $(X'X)\hat{\beta} = X'Y$ When the rank of the matrix $X'X$ is not full, an unrestrictive solution is possible with the 'solution with generalized inverse matrices' method.

In our study, equations were established based on tabular data to be used in two-factor cross-classification models, the first factor being doctors (public and special) and the second factor being the social status of patients.

- The factors that can provoke confrontations (or violence) faced by doctors include:
- Long waiting times,
- The excessive demands of patients' relatives,
- Low education levels,
- Stress caused by nagging or rancour of patients' relatives,
- Crowded and noisy workplace settings,
- Long, tiring working hours,
- Communication problems, leading to misunderstandings,
- Insufficient staff and tiredness,
- Insufficient security guard and police support,
- Inadequate crisis management.

In addition, physical violence–such as armed assault, lethal force, stabbing and verbal abuse arising from crowded and noisy workplaces was also observed.

The model seeks to determine which psychological factors drive patients and/or their relatives to resort to confrontation or violence. The ten issues listed above were tested to identify the ones most likely, with a given probability.

The two-factored linear model is given by:

$$y_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij} \qquad (1)$$

Where the first factor is denoted as A, and the second as B,

α_i: is the effect of the ith level of factor A

β_j: is the effect of the jth level of factor B

μ: is the general average

ε_{ij}: is the error term

y_{ij}: is the effect on ith row and jth column

Then, levels of α-factor and b levels of β-factor are subscripted like below:

i = 1, 2, 3,...,a, and j = 1, 2, 3,…,b

In our example, a = 2, b = 10. The term (a, b) shows the number of the observations in the cells.

The matrix form of equation (1) is $Y = X\beta + \varepsilon$ and the corresponding normal equation is $(X'X)\hat{\beta} = X'Y$. This can be solved with g-inverse. The best estimator will be determined from these results using a single solution vector. Here, $\hat{\beta}' = \{\mu, \alpha_r, \beta_c\}$, (where r: rows, c: columns) is the vector of unknowns, $\hat{\beta}$ which consists of three sub-vectors. It can be written as

$\hat{\beta}' = \{\mu, \alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_5, \beta_6, \beta_7, \beta_8, \beta_9, \beta_{10}\}$ in open form.

In statistical research, unknown model parameters are estimated based on observation or experimentation, and conclusions are drawn by statistical testing [1]. When using qualitative variables, Graybill [2-4], [7] termed these models general linear models. Given that quantitative variables were used in our study, the term random effects model best describes the model utilized [9-11], [13-15], [16],[17]. A similar paper is UNICEF's a statistical analysis of violence against children that includes only statistical data [18]. Another is for violence of women in India [19].

2. Materials and Methods

Linear models are appropriate for estimating data derived from case studies. They are easy to compute and solve. Non-linear models can only be useful by transforming them to linear ones by means of specific methodology.

Table 1. Twenty observations about factors involved in violence with doctors

	Factors											
	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}	$y_{.j}$	$y_{.j}$
α	7.2	7.5	7.9	8.7	6.8	8.2	7.4	7.7	8.6	7.3	77.3	148.13
	8	4	6	10	6	8	5	6	6	5	64	−0.61
$y_{i.}$	15.2	11.5	13.9	18.7	12.8	16.2	12.4	13.7	14.6	12.3	141.3	$y_{.}$ = 147.52

α: The factor of the institutions where the doctors work

β_1: Long waiting times

β_2: Excessive demands of patient and relatives

β_3: Low education levels

β_4: Stressed patient relatives

β_5: Crowded and noisy settings

β_6: Long working times

β_7: Communication problems like misunderstanding

β_8: Staff insufficiency and tiredness

β_9: Insufficient security and police support

β_{10}: Insufficiency in crisis management

The data in Table 1. are a non-interactive 2-way crossed classification linear model that can form a system of equations.

As seen in the example, there is a model in which the two factors are cross- classified. The data shows the points, which are given out of 10. These are the average values about the points provided by 75 doctors in 2013. If we write the equation (1) of the data in Table 1. clearly, it is:

$$\begin{aligned}
7.2 &= y_{11} = \mu + \alpha_1 + \beta_1 + \varepsilon_{11} \\
7.5 &= y_{12} = \mu + \alpha_1 + \beta_2 + \varepsilon_{12} \\
7.9 &= y_{13} = \mu + \alpha_1 + \beta_3 + \varepsilon_{13} \\
8.7 &= y_{14} = \mu + \alpha_1 + \beta_4 + \varepsilon_{14} \\
6.8 &= y_{15} = \mu + \alpha_1 + \beta_5 + \varepsilon_{15} \\
8.2 &= y_{16} = \mu + \alpha_1 + \beta_6 + \varepsilon_{16} \\
7.4 &= y_{17} = \mu + \alpha_1 + \beta_7 + \varepsilon_{17} \\
7.7 &= y_{18} = \mu + \alpha_1 + \beta_8 + \varepsilon_{18} \\
8.6 &= y_{19} = \mu + \alpha_1 + \beta_9 + \varepsilon_{19} \\
7.3 &= y_{110} = \mu + \alpha_1 + \beta_{10} + \varepsilon_{110} \\
8 &= y_{21} = \mu + \alpha_2 + \beta_1 + \varepsilon_{21} \\
4 &= y_{22} = \mu + \alpha_2 + \beta_2 + \varepsilon_{22} \\
6 &= y_{23} = \mu + \alpha_2 + \beta_3 + \varepsilon_{23} \\
10 &= y_{24} = \mu + \alpha_2 + \beta_4 + \varepsilon_{24} \\
6 &= y_{25} = \mu + \alpha_2 + \beta_5 + \varepsilon_{25} \\
8 &= y_{26} = \mu + \alpha_2 + \beta_6 + \varepsilon_{26}
\end{aligned} \qquad (2)$$

$$5 = y_{27} = \mu + \alpha_2 + \beta_7 + \varepsilon_{27}$$
$$6 = y_{28} = \mu + \alpha_2 + \beta_8 + \varepsilon_{28}$$
$$6 = y_{29} = \mu + \alpha_2 + \beta_9 + \varepsilon_{29}$$
$$5 = y_{210} = \mu + \alpha_2 + \beta_{10} + \varepsilon_{210}$$

For simplicity in the computation of the system of equations, we express the system of equations (2) with matrices. Thus we can write:

$$\begin{pmatrix} 7.2 \\ 7.5 \\ 7.9 \\ 8.7 \\ 6.8 \\ 8.2 \\ 7.4 \\ 7.7 \\ 8.6 \\ 7.3 \\ 8 \\ 4 \\ 6 \\ 10 \\ 6 \\ 8 \\ 5 \\ 6 \\ 6 \\ 5 \end{pmatrix} = \begin{pmatrix} y_{11} \\ y_{12} \\ y_{13} \\ y_{14} \\ y_{15} \\ y_{16} \\ y_{17} \\ y_{18} \\ y_{19} \\ y_{110} \\ y_{21} \\ y_{22} \\ y_{23} \\ y_{24} \\ y_{25} \\ y_{26} \\ y_{27} \\ y_{28} \\ y_{29} \\ y_{210} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \\ \beta_7 \\ \beta_8 \\ \beta_9 \\ \beta_{10} \end{pmatrix} + \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{14} \\ \varepsilon_{15} \\ \varepsilon_{16} \\ \varepsilon_{17} \\ \varepsilon_{18} \\ \varepsilon_{19} \\ \varepsilon_{110} \\ \varepsilon_{21} \\ \varepsilon_{22} \\ \varepsilon_{23} \\ \varepsilon_{24} \\ \varepsilon_{25} \\ \varepsilon_{26} \\ \varepsilon_{27} \\ \varepsilon_{28} \\ \varepsilon_{29} \\ \varepsilon_{210} \end{pmatrix} \quad (3)$$

Figure 1. Equation (3)

In this 2-factor cross-classification model

Y: Vector of observations, size (20×1),

X: the matrix of coefficients which consists of 0 s and 1 s, size (20×13),

$\beta = \beta' = (\mu\ \alpha_i\ \beta_j)$ the vector of unknown parameters, which consists of 3 sub-vectors, size (13×1)

ε : vector of errors, size (13 × 1)

2.1. The Matrix X and Its Properties

1) It consists of only 0s and 1s.

2) The 1st column is equal to the sum of both the 2nd and 3rd columns and to the sum of the 4th-13th columns. Thus, its column rank is not full.

2.2. Models with Full or Not Full Rank

Like variance analysis (VA) models, the models in which the independent variables are generally qualitative are called Models with not full rank and the models in which the independent variables are quantitative are called Models with full rank.

In the models with full rank where the normal equations are like $(X'X)\hat{\beta} = X'y$, because the inverse [$\det(X'X) \neq 0$]can be computed because the matrix $X'X$ is not singular, $\hat{\beta}$ has a single solution like [3, 4]

$$\hat{\beta} = (X'X)^{-1} X'y \quad (4)$$

In the qualitative variable models with not full rank, we have a problem estimating β because the matrix X is not irreversible [5],[6],[8]. In that case there are two approaches:

a) re-parametering,
b) g-inverse.

In our model, because the matrix X is not full rank (1st column is equal to the sum of both the 2nd and 3rd columns, and the 4th-13th columns) the number of linear relations between columns (or rows) causes the matrix $X'X$ not to be of full rank. That is, the degree of rank decreases by 2. Because $r(X) = r(X'X)$ when x is not of full rank, the symmetric matrix $X'X$ cannot be of full rank. In classical meaning, it is not irreversible. That is, either there does not exist a $\hat{\beta}$ vector or there are infinite $\hat{\beta}$ vectors. Furthermore, the linear relationship between the rows of the $X'X$ matrix exists between the elements of the $X'y$ vector if the normal equations are consistent and the system has more than one solution only in the case of consistency. It is obvious from (3) that the most appropriate solution to these comments can be made with g-inverse [4].

Let us construct the matrices X' and $X'X$ for our problem:

$$X' = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Figure 2 : X' matrix

$$X'X = \begin{pmatrix} 20 & 10 & 10 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 10 & 10 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 10 & 0 & 10 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} \quad (5)$$

Figure 3 $X'X$ matrix

In the symmetric $X'X$ matrix of size 13×13, the 1st row (or column) is equal to the sum of the 2nd, 3rd rows (or columns) and to the sum of the 4th-13th rows (or columns). As seen, this linear relationship is peculiar to the matrix $X'X$

When written in general form:

We can write the matrix $X'X$ like above, where $n_{..}$ shows the total number of observations, $n_{.a}$ shows the α_1, α_2 levels (crisis period) which are the same for each level, $n_{b.}$ shows the β_1, β_2, β_3, β_4, β_5 levels which are the same for each level, and n_{ba} shows the number of observations for each union α_a, β_b:

$$X'X = \begin{pmatrix} n_{..} & n_{.a} & n_{.a} & n_{b.} & n_{b.} & n_{b.} & n_{b.} & n_{b.} & n_{b.} & n_{b.} & n_{b.} & n_{b.} & n_{b.} \\ n_{.a} & n_{.a} & 0 & n_{ba} & n_{ba} & n_{ba} & n_{ba} & n_{ba} & n_{ba} & n_{ba} & n_{ba} & n_{ba} & n_{ba} \\ n_{.a} & 0 & n_{.a} & n_{ba} & n_{ba} & n_{ba} & n_{ba} & n_{ba} & n_{ba} & n_{ba} & n_{ba} & n_{ba} & n_{ba} \\ n_{b.} & n_{ba} & n_{ba} & n_{b.} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ n_{b.} & n_{ba} & n_{ba} & 0 & n_{b.} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ n_{b.} & n_{ba} & n_{ba} & 0 & 0 & n_{b.} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ n_{b.} & n_{ba} & n_{ba} & 0 & 0 & 0 & n_{b.} & 0 & 0 & 0 & 0 & 0 & 0 \\ n_{b.} & n_{ba} & n_{ba} & 0 & 0 & 0 & 0 & n_{b.} & 0 & 0 & 0 & 0 & 0 \\ n_{b.} & n_{ba} & n_{ba} & 0 & 0 & 0 & 0 & 0 & n_{b.} & 0 & 0 & 0 & 0 \\ n_{b.} & n_{ba} & n_{ba} & 0 & 0 & 0 & 0 & 0 & 0 & n_{b.} & 0 & 0 & 0 \\ n_{b.} & n_{ba} & n_{ba} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & n_{b.} & 0 & 0 \\ n_{b.} & n_{ba} & n_{ba} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & n_{b.} & 0 \\ n_{b.} & n_{ba} & n_{ba} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & n_{b.} \end{pmatrix} \quad (6)$$

Figure 4 : general form of $X'X$ matrix

or more generally we can write:

$$X'X = \begin{pmatrix} n_{..} & n_{.1} & n_{2} & n_{1.} & n_{2.} & n_{3.} & n_{4.} & n_{5.} & n_{6.} & n_{7.} & n_{8.} & n_{9.} & n_{10.} \\ n_{.1} & n_{.1} & 0 & n_{11} & n_{21} & n_{31} & n_{41} & n_{51} & n_{61} & n_{71} & n_{81} & n_{91} & n_{101} \\ n_{2} & 0 & n_{2} & n_{12} & n_{22} & n_{32} & n_{42} & n_{52} & n_{62} & n_{72} & n_{82} & n_{92} & n_{102} \\ n_{1.} & n_{11} & n_{12} & n_{1.} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ n_{2.} & n_{21} & n_{22} & 0 & n_{2.} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ n_{3.} & n_{31} & n_{32} & 0 & 0 & n_{3.} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ n_{4.} & n_{41} & n_{42} & 0 & 0 & 0 & n_{4.} & 0 & 0 & 0 & 0 & 0 & 0 \\ n_{5.} & n_{51} & n_{52} & 0 & 0 & 0 & 0 & n_{5.} & 0 & 0 & 0 & 0 & 0 \\ n_{6.} & n_{61} & n_{62} & 0 & 0 & 0 & 0 & 0 & n_{6.} & 0 & 0 & 0 & 0 \\ n_{7.} & n_{71} & n_{72} & 0 & 0 & 0 & 0 & 0 & 0 & n_{7.} & 0 & 0 & 0 \\ n_{8.} & n_{81} & n_{82} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & n_{8.} & 0 & 0 \\ n_{9.} & n_{91} & n_{92} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & n_{9.} & 0 \\ n_{10.} & n_{101} & n_{102} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & n_{10.} \end{pmatrix} \quad (7)$$

Figure 5 : more general form of $X'X$ matrix

Now, we write the equations clearly with symbols:

$$20 = n_{..} = \sum n_{.a} = n_{.1} + n_{.2} = 10 + 10,$$
$$20 = n_{..} = \sum n_{b.}$$
$$= n_{1.} + n_{2.} + n_{3.} + n_{4.} + n_{5.} + n_{6.} + n_{7.} + n_{8.} + n_{9.} + n_{10.}$$
$$= 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 = 20$$

The vector $X'y$ can be found as below:

$$X'y = \begin{pmatrix} 141.3 \\ 77.3 \\ 64 \\ 15.2 \\ 11.5 \\ 13.9 \\ 18.7 \\ 12.8 \\ 16.2 \\ 12.4 \\ 13.7 \\ 14.6 \\ 12.3 \end{pmatrix} = \begin{pmatrix} y_{..} \\ y_{.1} \\ y_{.2} \\ y_{1.} \\ y_{2.} \\ y_{3.} \\ y_{4.} \\ y_{5.} \\ y_{6.} \\ y_{7.} \\ y_{8.} \\ y_{9.} \\ y_{10.} \end{pmatrix}$$

It is obvious that the same relationship between the rows of the matrix $X'X$ is also present between the elements of the vector $X'y$. Because the system of equations $Y = X\beta + \varepsilon$ is consistent, there are infinite solutions for $\hat{\beta}$. Graybill and Pringle used a method that they named transformation and re-parametering to transform the models with not full rank to models with full rank. The analysis can be made easily without applying some restrictions (e.g., $\sum \alpha_i = 0$) like classic analyses and it does not matter which g-inverse and which $\hat{\beta}$ solution are there. This comes from invariability and depends on the estimable function properties of our equations. We will not give theorems that show that the solution will be uniquely independent of which one of the infinite solutions of the g-inverse is used in our study [4]. Now we will give normal equations based on Table 1. and we will make our statistical application. We know that a solution of the normal equations $X'X\hat{\beta} = X'y$ is $\hat{\beta} = GX'y$ and the second solution is like $\hat{\beta} = GX'y + (H - I)Z$. Here, $X'XGX'X = X'X$ and $H = GX'X$ with Z is a random vector of unknowns. Let us describe the method that we will use to compute g-inverse. In the models with not full rank, the matrix X´X is a random coefficient matrix and replaces A in the system of equations: $Y = AX$ and it is symmetric. Now let us talk about a simple but useful method for symmetric matrices.

With this method, the computation of g-inverse fully depends on knowledge of the rank of the matrix A. We choose an (r x r) matrix that is not singular and is the largest sub-matrix A [4]. We denote this with A_{11} and let the other parts of A be:

$$A_{mxn} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad (8)$$

It is possible to make changes on the rows and the columns of A to make the rank maximum, and A_{11} is not required to be unique.

$$G_{nxm} = \begin{pmatrix} A^{-1}{}_{11} & O \\ O & O \end{pmatrix} \quad (9)$$

is a g-inverse of the matrix A. If AG and GA are computed, it is seen that AGA = G.

2.3. *G inverse computations*

1. a r rank(degree) of non-singular co-factor of a symmetric matrix A is found.
2. M^{-1} is established.
3. The transpose of M^{-1} is established. That is, $(M^{-1})'$ is found.
4. For each element for which the co-factor M is established in matrix A, the corresponding element is placed in $(M^{-1})'$ and the other elements are zeros.
5. The obtained matrix is G, the generalized inverse matrix of A.

Now let us consider the XX' matrix (5) which was established due to Table 1. The degree of the $X'X$ indicator matrix is 13. The sum of the 2nd and the 3rd rows give the 1st row and the sum of the 4th-13th rows (columns) give the 2nd row (column).

Because of this linear relation $rank(XX') = 13 - 2 = 11$. We denote the total number of levels for *α-factor* with **a** and if the level number about the *β-factor* is **b**,

$$Rank = r = r(X'X) = 1 + a + b - 2 = a + b - 1 \quad (10)$$

[Note: The number 2 denotes the number of the linear relationships]

We know that a = 2, b = 10 from Table 1. According to this,

$$r = r(X'X) = 2 + 10 - 1 = 11 \quad (11)$$

Because the number 1 in equation (10) corresponds to μ (the population average) and the degree of the matrix $X'X$ is computed by $m = 1 + p + q$, we find that $r(X'X) = 1 + p + q - (1 + p) = q$.

2.4. *Normal Equations*

The matrix equivalent of equation (3) is a matrix equation like $Y = X\beta + \varepsilon$. We will write the appropriate normal equations to this. These are like $X'X\ \hat{\beta} = X'y$ or $(X'X)b^0 = X'y$. Similar to (3), the normal equations can be written like below:

$$\begin{pmatrix} \mu^0 & \alpha_1^0 & \alpha_2^0 & \beta_1^0 & \beta_2^0 & \beta_3^0 & \beta_4^0 & \beta_5^0 & \beta_6^0 & \beta_7^0 & \beta_8^0 & \beta_9^0 & \beta_{10}^0 \\ 20 & 10 & 10 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 10 & 10 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 10 & 0 & 10 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \mu^0 \\ \alpha_1^0 \\ \alpha_2^0 \\ \beta_1^0 \\ \beta_2^0 \\ \beta_3^0 \\ \beta_4^0 \\ \beta_5^0 \\ \beta_6^0 \\ \beta_7^0 \\ \beta_8^0 \\ \beta_9^0 \\ \beta_{10}^0 \end{pmatrix} = \begin{pmatrix} y_{..} \\ y_1 \\ y_2 \\ y_{1.} \\ y_{2.} \\ y_{3.} \\ y_{4.} \\ y_{5.} \\ y_{6.} \\ y_{7.} \\ y_{8.} \\ y_{9.} \\ y_{10.} \end{pmatrix} = \begin{pmatrix} 141.3 \\ 77.3 \\ 64 \\ 15.2 \\ 11.5 \\ 13.9 \\ 18.7 \\ 12.8 \\ 16.2 \\ 12.4 \\ 13.7 \\ 14.6 \\ 12.3 \end{pmatrix} \quad (12)$$

Figure 6 :normal equations

Because $rank(X'X) = 11$ from (11); from (4.2) we chose a non-singular signed minor whose degree is 11 from $X'X$. We denote this matrix as M:

$$M = \begin{pmatrix} 10 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$$\det(M) = 5120 \neq 0$$

$$M^{-1} = \begin{pmatrix} 0.2 & -0.1 & -0.1 & -0.1 & -0.1 & -0.1 & -0.1 & -0.1 & -0.1 & -0.1 & -0.1 \\ -0.1 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 \\ -0.1 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 \\ -0.1 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 \\ -0.1 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 \\ -0.1 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 \\ -0.1 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 \\ -0.1 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 \\ -0.1 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 \\ -0.1 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 \\ -0.1 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 \end{pmatrix}$$

Figure 7 :The inverse of M matrix

and the matrix is symmetric. That is, $(M^{-1})' = M^{-1}$. If we write G (g-inverse) according to the rules in (2.4), we find:

$$G = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & -0.1 & -0.1 & -0.1 & -0.1 & -0.1 & -0.1 & -0.1 & -0.1 & -0.1 & -0.1 \\ 0 & 0 & -0.1 & 0.55 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 \\ 0 & 0 & -0.1 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 \\ 0 & 0 & -0.1 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 \\ 0 & 0 & -0.1 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 \\ 0 & 0 & -0.1 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 \\ 0 & 0 & -0.1 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 \\ 0 & 0 & -0.1 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 \\ 0 & 0 & -0.1 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 \\ 0 & 0 & -0.1 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 \\ 0 & 0 & -0.1 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 \end{pmatrix}$$

Figure 8 :G matrix

If a < b or a > b then $\mu^0 = 0$ and $\alpha_1^0 = 0$ or $\beta_b^0 = 0$. Because a = 2, b = 10 and a < b (that is $2 < 10$) in our study, it is possible to obtain a simpler solution by zeroing 1 + a levels. Thus, the solution will be like $\beta^{0''} = [0'_{1x(1xa)} \ \bar{Y}']$. Now we show that $X'XGX'X = X'X$:

$$X'XG = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$X'X = \begin{pmatrix} 20 & 10 & 10 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 10 & 10 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 10 & 0 & 10 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

we find:

$$X'XGX'X = \begin{pmatrix} 20 & 10 & 10 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 10 & 10 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 10 & 0 & 10 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$$= X'X$$

Because $\hat{\beta} = X'y$, the right side of the matrix equation is:

$$\hat{\beta} = X'y = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & -0.1 & -0.1 & -0.1 & -0.1 & -0.1 & -0.1 & -0.1 & -0.1 & -0.1 & -0.1 \\ 0 & 0 & -0.1 & 0.55 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 \\ 0 & 0 & -0.1 & 0.05 & 0.55 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 \\ 0 & 0 & -0.1 & 0.05 & 0.05 & 0.55 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 \\ 0 & 0 & -0.1 & 0.05 & 0.05 & 0.05 & 0.55 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 \\ 0 & 0 & -0.1 & 0.05 & 0.05 & 0.05 & 0.05 & 0.55 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 \\ 0 & 0 & -0.1 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.55 & 0.05 & 0.05 & 0.05 & 0.05 \\ 0 & 0 & -0.1 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.55 & 0.05 & 0.05 & 0.05 \\ 0 & 0 & -0.1 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.55 & 0.05 & 0.05 \\ 0 & 0 & -0.1 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.55 & 0.05 \\ 0 & 0 & -0.1 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.55 \end{pmatrix} \begin{pmatrix} 141.3 \\ 77.3 \\ 64.0 \\ 15.2 \\ 11.5 \\ 13.9 \\ 18.7 \\ 12.8 \\ 16.2 \\ 12.4 \\ 13.7 \\ 14.6 \\ 12.3 \end{pmatrix} = \begin{pmatrix} 0.00 \\ 0.00 \\ -1.33 \\ 8.27 \\ 6.42 \\ 7.52 \\ 10.02 \\ 7.07 \\ 8.78 \\ 6.87 \\ 7.52 \\ 7.97 \\ 6.82 \end{pmatrix}$$

Figure 9 : $\hat{\beta}$ vector calculation

$\hat{\beta}$ can also be made with the computation of another g-inverse. The fact that there is not a unique g-inverse is not undesirable. It brings an important approach of the invariability of some results regardless of which solution is found. We show this important property of G:

We find the matrix H from $XGX'X = y$ or $XH = X$ and the solution of the system of equations $AX = Y$ is $\hat{\beta} = GX'y + (H - I)Z$.

Z is a vector of (H-I) degrees or row degrees of $GX'X$.

Now we return to our example. Because $H = GX'X$ from (4) and if Z is any solution vector, we find:

$$H = GX'X = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&-1&1&0&0&0&0&0&0&0&0&0&0 \\ 1&1&0&1&0&0&0&0&0&0&0&0&0 \\ 1&1&0&0&1&0&0&0&0&0&0&0&0 \\ 1&1&0&0&0&1&0&0&0&0&0&0&0 \\ 1&1&0&0&0&0&1&0&0&0&0&0&0 \\ 1&1&0&0&0&0&0&1&0&0&0&0&0 \\ 1&1&0&0&0&0&0&0&1&0&0&0&0 \\ 1&1&0&0&0&0&0&0&0&1&0&0&0 \\ 1&1&0&0&0&0&0&0&0&0&1&0&0 \\ 1&1&0&0&0&0&0&0&0&0&0&1&0 \\ 1&1&0&0&0&0&0&0&0&0&0&0&1 \end{pmatrix}$$

Let one of the solutions be like $Z' = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

$\hat{\beta}_{01} = GX'y + (H - I)Z$ and $(H - I)Z = 0$. Thus, we find:

$$\hat{\beta}_{01} = X'y = \begin{pmatrix} 0.00 \\ 0.00 \\ -1.33 \\ 8.27 \\ 6.42 \\ 7.52 \\ 10.02 \\ 7.07 \\ 8.78 \\ 6.87 \\ 7.52 \\ 7.97 \\ 6.82 \end{pmatrix}$$

This is the same as $\hat{\beta}'$.

If we randomly take $Z' = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$ for the second solution,

$$\hat{\beta}_{02} = \begin{pmatrix} 0.00 \\ 0.00 \\ -1.33 \\ 8.27 \\ 6.42 \\ 7.52 \\ 10.02 \\ 7.07 \\ 8.78 \\ 6.87 \\ 7.52 \\ 7.97 \\ 6.82 \end{pmatrix} + \left[\begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&-1&1&0&0&0&0&0&0&0&0&0&0 \\ 1&1&0&1&0&0&0&0&0&0&0&0&0 \\ 1&1&0&0&1&0&0&0&0&0&0&0&0 \\ 1&1&0&0&0&1&0&0&0&0&0&0&0 \\ 1&1&0&0&0&0&1&0&0&0&0&0&0 \\ 1&1&0&0&0&0&0&1&0&0&0&0&0 \\ 1&1&0&0&0&0&0&0&1&0&0&0&0 \\ 1&1&0&0&0&0&0&0&0&1&0&0&0 \\ 1&1&0&0&0&0&0&0&0&0&1&0&0 \\ 1&1&0&0&0&0&0&0&0&0&0&1&0 \\ 1&1&0&0&0&0&0&0&0&0&0&0&1 \end{pmatrix} - \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1 \end{pmatrix} \right] \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Figure 10 : $\hat{\beta}_{02}$ vector calculation

$$= \begin{bmatrix} 0 \\ 0 \\ -1.33 \\ 8.27 \\ 6.42 \\ 7.62 \\ 10.02 \\ 7.07 \\ 8.77 \\ 6.87 \\ 7.52 \\ 7.97 \\ 6.82 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ -1 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -2.33 \\ 10.27 \\ 8.42 \\ 9.62 \\ 12.02 \\ 9.07 \\ 10.77 \\ 8.87 \\ 9.52 \\ 9.97 \\ 8.82 \end{bmatrix}$$

or

$$\hat{\beta}'_{02} = \begin{bmatrix} -1 & -1 & -2.33 & 10.27 & 8.42 & 9.62 & 12.02 & 9.07 & 10.77 & 8.87 & 9.52 & 9.97 & 8.82 \end{bmatrix}$$

.

We have found a solution different from $\hat{\beta}_{01}$. Now we show that regardless of whatever solution we take, the solution is invariable. Thus we find:

$$\hat{\beta}'_{01}X'y = \begin{bmatrix} 0 & 0 & -1.33 & 8.27 & 6.42 & 7.62 & 10.02 & 7.07 & 8.77 & 6.87 & 7.52 & 7.97 & 6.82 \end{bmatrix} \begin{pmatrix} 141.3 \\ 77.3 \\ 64.0 \\ 15.2 \\ 11.5 \\ 13.9 \\ 18.7 \\ 12.8 \\ 16.2 \\ 12.4 \\ 13.7 \\ 14.6 \\ 12.3 \end{pmatrix} = 1028.781$$

and

$$\hat{\beta}'_{02}X'y = \begin{bmatrix} -1 & -1 & -2.33 & 10.27 & 8.42 & 9.62 & 12.02 & 9.07 & 10.77 & 8.87 & 9.52 & 9.97 & 8.82 \end{bmatrix} \begin{pmatrix} 141.3 \\ 77.3 \\ 64.0 \\ 15.2 \\ 11.5 \\ 13.9 \\ 18.7 \\ 12.8 \\ 16.2 \\ 12.4 \\ 13.7 \\ 14.6 \\ 12.3 \end{pmatrix} = 1028.781$$

If we take the rounding into account, it is obvious that the two $\hat{\beta}$ solutions are equal. Despite that, the solutions are within 0.001 of each other. This difference is neglectable. Thus, it shows that regardless of whatever solution of $\hat{\beta}'_0 X'y$ we take, the result is invariable.

2.5. Non-interactive 2-Factored Cross-Classified Variance Analysis

We have written the non-interactive 2-factor variance analysis model like below:

$$y_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij}$$

(i = 1(1) k; j = 1(1) l)

Here, μ represents the general average (population average), α_i is the effect of the ith level (of factor A), β_j is the effect of the jth level (of factor B). Because there is not a parameter (like γ_{ij}) that gives the effect of two parameters together in this model, the model is called non-interactive. In this model, the ε_{ij} s are assumed to have a normal distribution with a 0 average $\left(E(\varepsilon_{ij})=0\right)$ and σ_ε^2 variance.

2.6. The Hypothesis Tests

We can write the hypothesis tests that are appropriate for our real data in the table as below. Because there are two factors, two H_0 hypotheses must be suggested [7],[12]. Let us write the hypotheses in which the effects of the first and the second factors are all zero:

H_0 : $\alpha_i = 0$ (i = 1 , 2,……k)

H_a : $\alpha_i \neq 0$ (for at least one value of i) (13)

H_0 : $\beta_j = 0$ (j = 1 , 2,……l)

H_a : $\beta_j \neq 0$ (for at least one value of j)

If $F_{\alpha 1} \geq F_\alpha$ or $F_{SSFA} \geq F_\alpha$ then H_0 will be rejected. Similarly, if we write for the second factor, H_0 again will be rejected if $F_{\beta j} \geq F_\alpha$ or $F_{SSFB} \geq F_\alpha$, where α is the significance level, not α_i. We will explain the SSFA and SSFB below.

2.7. Analysis of the Sum of Squares

Let GSS denotes the general sum of squares, SSFA denotes the sum of squares of factor A, SSFB denotes the sum of squares of factor B, and SSE denote the sum of squares of errors from residuals. If we write these three expressions as a sum:

GSS = SSE + SSFA + SSFB

Now, we will write the chi-square ($\chi 2$) values and the related degrees of freedom, and we will compute $F_{\alpha i}$ and $F_{\beta j}$ Fisher distributions from these. The two-factored variance analysis table that is established based on this information is shown in table 2. and table 3. below:

Table 2: Variance Analysis

Source of variation	Degrees of freedom	Sum of Square	Average of Square	Statistics of F
Factor I (A)	$k-1$	$\sum_i^k (\bar{y}_{.j} - \bar{y}_{..})^2 \, SSFA = l$	$FAA = \frac{SSFA}{k-1}$	$F_{SSFA} = \frac{FAA}{KA}$
Factor I (B)	$l-1$	$\sum_j^l (\bar{y}_{i.} - \bar{y})^2 SSFB = k$	$FBA = \frac{SSFB}{l-1}$	$F_{SSFB} = \frac{FBA}{KA}$
Errors~ residuals	$(k-1)(l-1)$	$SSE = \sum_i^k \sum_j^l (y_{ij} - \bar{y}_{i.} - \bar{y}_{.j} + \bar{y})^2$	$KA = \frac{SSE}{(k-1)(l-1)}$	
Sum of variation	$lk-1$	$GSS = \sum_i^k \sum_j^l (y_{ij} - \bar{y}_{..})^2$		

Table 3: Variance Analysis with Numerical Values

Source of variation	Degrees of freedom	Sum of Square	Average of Square	Statistics of F
Factor I (A)	k – 1 = 2 – 1 = 1	SSFA = 8.8445	FAA = 8.8445/1 = 8.8445	F_{SSFA} = FAA/KA = 0.107 ≅ 0.1
Factor II (B)	l – 1 = 10 – 1 = 9	SSFB = 20.9005	FBA = 20.9005/9 = 2.32227	F_{SSFB} = FBA/KA = 0.028 ≅ 0.03
Errors~ residuals	(k – 1)(l – 1) = 9	SSE = 743.2795	KA = 743.2795/9 = 82.586	
Sum of variation	l k – 1 = 19	GSS = 773.0245		

3. Results and Discussion

We compared the F distribution values in Table 3. with real F-table values and came to a conclusion about the hypothesis in (18).

At the 5% confidence level, the F theoretical values are found to be $F_{1,9,0.05} = 5.12$ and $F_{9,9,0.05} = 3.18$ with (1, 9) and (9, 9) degrees of freedom, respectively.

The F-statistics that are computed from Table 3. are $F_{SSFA} = 0.1$ and $F_{SSF} = 0.03$. Thus, because the theoretical values are greater than the real F values the H_0 hypotheses that are established for both factors cannot be rejected. Therefore, the conclusion is the model $E(y_{ij}) = \mu + \alpha_i + \beta_j$ can explain the changes in y more than the model E(y_{ij}) = μ (the model that only includes the average).

As seen in the solution vector $\hat{\beta}'$, the fact that one of the α_i factor levels $\alpha_1 = 0$ causes the significance level to be concentrated on the second factor β_j. Now let us look at the solution vector again and try to interpret the levels according to their significance levels:

$$\hat{\beta}' = [\mu\ \alpha_1\ \alpha_2\ \beta_1\ \beta_2\ \beta_3\ \beta_4\ \beta_5\ \beta_6\ \beta_7\ \beta_8\ \beta_9\ \beta_{10}]$$
$$= [0.00\ 0.00\ -1.33\ 8.27\ 6.42\ 7.52\ 10.02\ 7.07\ 8.78\ 6.87\ 7.52\ 7.97\ 6.82]$$

Because $\alpha_1 = 0$ "the public institutions" have no effect. The private institutions have a negative effect no matter which solution vector is taken ($\alpha_2 = -1.33$) and the most important factor predicting violence is $\beta_4 = 10.02$, that is, "the stressed patient relatives" have the highest effect. In the $\hat{\beta}' = \hat{\beta}'_{01}$ solution vector its value is 10.02% and in the $\hat{\beta}'_{02}$ solution vector the same variable is again found at the highest rate (12.02%). The 2nd most important variable "long working times" is $\beta_6 = 8.78\%$ in $\hat{\beta}'_{01}$ and it is 10.78% in $\hat{\beta}'_{02}$. The 3rd most important variable is β_1 (long waiting times). It is 8.27% in $\hat{\beta}'_{01}$ and 8.42% in $\hat{\beta}'_{02}$.

Thus, because of the special structure of the matrix X [it consists of 0s and 1s], the solution is obtained by showing a special type of these qualitative variables using a type of regression model. The system, which consists of qualitative variables and normal equations, is irreversible because it is not a model with full rank. We can estimate $\beta = \hat{\beta}'$ from $G = (X'X)^{-1}$ without applying any restrictions if a procedure like the solution of the models with full rank is applied.

4. Conclusions

We can conclude that it is not enough for the average to explain the model singularly.

Another conclusion is that the patients' βj factor must be included in the model with all of its factors. It was found that G-inverse and Factor analysis methods which are introduced in this study are superior to other published techniques.

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